Numerical Methods for Time-Dependent PDEs

Example questions for an Exam (Tentamen): WISL602

May 2024

It will be a <u>closed book exam</u>.

Check the exercises from lectures 1-12, as listed on the webpage, as well!

Normally, (around) 8 questions will be asked in a typical exam.

Question 1

METHOD OF UNDETERMINED COEFFICIENTS
(a) Use the *method of undetermined coefficients* to derive a *fourth-order* accurate central approximation of u_x at the grid point $x_j = j\Delta x,\ j = 0, 1, ..., J$ with $\Delta x = \frac{1}{J}$ of the form (find the constants A and B):

$$\frac{-Au_{j+2} + Bu_{j+1} - Bu_{j-1} + Au_{j-2}}{\Delta x}.$$

(b) *Determine* the constants C and p in the error term: $C(\Delta x)^4 u^{(p)}(\xi)$.

Question 2

TIME-INTEGRATION/LOCAL TRUNCATION ERROR/STABILITY
(a) Consider the following method for the ODE $\dot{y} = f(t, y)$:

$$y^{n+1} = y^n + \Delta t\ [\frac{1}{7} f(t^{n+1}, y^{n+1}) + \frac{6}{7} f(t^n, y^n)].$$

Calculate the *local truncation error* τ for this method.

(b) Derive the *stability function*[1] $R(z)$ for the method in (a). Is the method *(un)conditionally stable*? Explain.

(c) Show that the *stability region* is bounded by a circle: *determine* α and r in:

$$[\Re(z) + \alpha]^2 + [\Im(z)]^2 = r^2.$$

[1] It is *convenient* to define $z = \lambda \Delta t$ with $z = \Re(z) + \Im(z)\mathbf{i}$ and $\mathbf{i} = \sqrt{-1}$.

Question 3

BOUNDARY LOCUS/STABILITY POLYNOMIAL/ZERO-STABLE
Consider the ODE: $\dot{y}(t) = f(y(t))$.

(a) Give the definition of the *boundary locus* and explain why it plays an important role in the numerical solution of ODEs.

(b) Let the following *six* functions be defined for $\varphi \in [0, 2\pi]$:

F1. $g_1(\varphi) = 1 - \mathrm{e}^{-\mathbf{i}\varphi}$, with $\mathbf{i} = \sqrt{-1}$.
F2. $g_2(\varphi) = \frac{3}{2} - 2\mathrm{e}^{-\mathbf{i}\varphi} + \frac{1}{2}\mathrm{e}^{-2\mathbf{i}\varphi}$.
F3. $g_3(\varphi) = \mathbf{i}\ \sin(\varphi)$.
F4. $g_4(\varphi) = \frac{3\mathbf{i}\sin(\varphi)}{\cos(\varphi)+2}$.
F5. $g_5(\varphi) = 1 + \mathrm{e}^{-\mathbf{i}\varphi} + \frac{1}{2}(1 - \mathrm{e}^{-\mathbf{i}\varphi})^2 + \frac{1}{3}(1 - \mathrm{e}^{-\mathbf{i}\varphi})^3$.
F6. $g_6(\varphi) = \mathrm{e}^{\mathbf{i}\varphi} - 1$.

and also the following *six* time-integration methods:

M1: Euler-Forward.
M2. BDF2 (an extension of Euler-Backward).
M3. Euler-Backward.
M4. explicit Midpoint.
M5. implicit Milne-Simpson: $y^{n+2} = y^n + \Delta t\ [\frac{1}{3}f(y^n) + \frac{4}{3}f(y^{n+1}) + \frac{1}{3}f(y^{n+2})]$.
M6. BDF3 (an extension of BDF2).

Associate each method M1, ..., M6 with one of the functions F1, .., F6 and explain your choice. What do the functions $g(\varphi) :\ [0, 2\pi] \to \mathbb{C}$ represent?

(c) *Sketch* the curves F1, F3 and F6 in the complex plane.

(d) Find the *stability polynomial* $\pi(z;\zeta)$ for the *3-step Adams-Bashforth* method:

$$y^{n+3} = y^{n+2} + \frac{\Delta t}{12}[5f(y^n) - 16f(y^{n+1}) + 23f(y^{n+2})].$$

(e) Consider the following two *linear multistep* methods:

$$3y^{n+2} = 4y^{n+1} - y^n + 2\Delta t f(y^n)$$

and

$$y^{n+2} = 4y^{n+1} - 3y^n - 2\Delta t f(y^n).$$

Verify whether these methods are *consistent* and *zero-stable*.[2]

[2]Hint: use properties of the *characteristic polynomials*: $\rho(\zeta)$ and $\sigma(\zeta)$.

Question 4

HEAT/ADVECTION
Consider the *advection* equation:

$$u_t + \gamma u_x = 0, \qquad u(x,0) = \mathrm{e}^{-x^2}, \quad x \in (-\infty, \infty), \quad \gamma \in \mathbb{R}.$$

(a) What is the *exact* solution $u(x,t)$ of this model?

(b) Show, using *Von Neumann* stability analysis, that the numerical scheme

$$u_j^{n+1} = (1 + \frac{\gamma \Delta t}{\Delta x}) u_j^n - \frac{\gamma \Delta t}{\Delta x} u_{j+1}^n$$

is always *<u>un</u>stable* for $\gamma = 1$ and *conditionally* stable for $\gamma = -1$.

(c) The method in (b) is *first-order accurate*. What happens for $\frac{\gamma \Delta t}{\Delta x} = -1$?

Question 5

FINITE ELEMENTS
Apply a *finite element method* with piecewise linear functions ϕ_j, both as approximating and test functions, to the PDE: $u_t = u_{xx} + f(t)$.

- Show that a *semi-discrete* ODE-system is obtained of the form:

$$\mathcal{A}\dot{\vec{u}}(t) + \mathcal{B}\vec{u}(t) = \vec{f}.$$

- Describe the *inner products* $\langle \cdot, \cdot \rangle$ appearing in the matrices $\mathcal{A}$ and $\mathcal{B}$.
- Which *names* are usually associated with the matrices $\mathcal{A}$ and $\mathcal{B}$?

Question 6

ADAPTIVE/NON-UNIFORM GRIDS
The *equidistribution* principle with monitor function $\omega > 0$ reads:

$$[x_\xi \omega]_\xi = 0, \quad x(0) = 0, \ x(1) = 1. \tag{1}$$

A *non-uniform grid* in the variable x is determined via the grid function $x(\xi)$ on a uniform grid distribution for ξ.

(a) Calculate the grid function $x(\xi)$ for $\omega = 1$. Which grid distribution follows from (1) in this case?

(b) Which property for the derivative of the transformation $\xi \to x$ ensures a *non-distorted* (= non-singular) non-uniform grid, i.e., a grid with $\Delta x_j > 0$, $\forall j$.

(c) Let the transformation $x(\xi) = 4\xi^3 + \xi^2 - 4\xi$ be given. Make clear that the property in part (b) is violated.

(d) Next, consider $x(\xi) = \xi^4$. Check that the property in part (b) is satisfied for this grid function.

(e) Define the *five* grid points $\xi_j = j\Delta\xi,\ j = 0, 1, ..., 4$. Sketch x from parts (c) and (d) as a function of $\xi \in [0, 1]$ and explain how you obtain the values of the non-uniform grid points x_j from these graphs (and mark those points).

Question 7

EXACT/NONSTANDARD FDs
(a) Verify that the scheme:

$$\left\{ \begin{array}{l} \frac{y^{n+1}-y^n}{1-\mathrm{e}^{-\Delta t}} = -y^n, \quad n = 0, 1, 2, ...; \quad \Delta t > 0, \\ \\ y^0 = 1, \quad \text{with } y^n \approx y(t^n) = y(n\Delta t) \end{array} \right\}$$

is an *exact* finite difference (FD) scheme for the ODE: $\left\{ \begin{array}{l} \dot{y}(t) = -y(t), \\ y(0) = 1. \end{array} \right\}$.

(b1) Which *denominator* function ϕ is being used in this case?
(b2) What is the essential property of ϕ in exact and nonstandard FDs?
(b3) Which ϕ is related to the Euler-Forward method?
(b4) Is the above scheme a *nonlocal* one? Explain!

Question 8

HAMILTONIAN PDE
Consider the time-dependent PDE: $u_t = u^q\ u_x - \epsilon\ u_{xxx}$ with $q \geq 2,\ \epsilon > 0$ and $u(x, t) \to 0$ as $|x| \to +\infty$.

(a) Calculate the *variational* derivative of the functional:

$$\mathcal{H} = \int_{-\infty}^{\infty} [\frac{\epsilon}{2}\ u_x^2 + \frac{1}{(q+1)(q+2)}\ u^{q+2}]\ \mathrm{d}x.$$

(b) Show that the PDE is a *Hamiltonian* PDE.

(c) Briefly[3] describe a *discretization* on the finite spatial interval $[-L, L]$ with $L \gg 1$ and a useful *time-integration* method for the Hamiltonian PDE in (b).

[3] For this answer, use (at most) three lines!

Question 9

TRAVELLING WAVES
Please check the exercises on TWs in lecture 11.

Question 10

CLASSIFICATION/METHOD OF LINES/TIME-INTEGRATION/STABILITY
Consider the *Telegraph* equation:

$$u_{tt} + \alpha u_t + \beta u = c^2 u_{xx}, \ \alpha \in \mathbb{R}, \ \beta \in \mathbb{R}, \ c \in \mathbb{R}. \tag{2}$$

(a) *Classify* PDE (2): is it elliptic, parabolic or hyperbolic?

(b) *Re-write* the PDE as a system of first-order PDEs in time, semi-discretize in space (the *first step* in the Method-of-Lines) and write the resulting ODE system[4] as:

$$\dot{\vec{\eta}} = M \ \vec{\eta}. \tag{3}$$

and *verify* that:

$$M = \begin{pmatrix} \mathcal{O} & \mathcal{I} \\ c^2 D_{2c} - \beta \mathcal{I} & -\alpha \mathcal{I} \end{pmatrix} \qquad \text{with} \qquad \vec{\eta} := [\vec{U}, \vec{V}]^T.$$

It can be derived that the eigenvalues μ of the matrix M satisfy[5]:

$$\mu^2 + \alpha\mu + \beta - c^2\lambda = 0,$$

where λ is an eigenvalue[6] of the matrix D_{2c}.

Next, consider in the *second step* of the Method-of-Lines, the following four time-integration methods:

1. Euler-Forward (EF).
2. Euler-Backward (EB).
3. (explicit) Midpoint method (MP).
4. Boundary-Value method (MP as underlying method with EB final condition).

(c) For the case $\alpha = \beta = 0$, $c = 1$: which method**s** would you recommend regarding their *stability properties* and which ones not? Motivate your choice!

(d) Same question as in part (c), but now for the case: $\alpha = 2$, $\beta = 0$, $c = 2$.

(e) Same question as in part (c) for the case: $\alpha = c = 0$, $\beta = -100$

[4] As you have learned during the lectures: the second-order three-point central approximation for u_{xx} at the grid point x_p can be represented by the matrix D_{2c}.
[5] You do not have to prove this.
[6] The eigenvalues of the matrix D_{2c} are: $\lambda_p = \frac{2}{(\Delta x)^2}(\cos(p\pi\Delta x) - 1), \ p = 1, 2, ..., M$.