10.1 (a)

Example 2.6 Consider the following problem in a domain $\Omega \subset R^2$:

(2.23a) $$-\mu \Delta u + \beta_1 \frac{\partial u}{\partial x_1} + \beta_2 \frac{\partial u}{\partial x_2} + u = f \quad \text{in } \Omega,$$

(2.23b) $$u = 0 \text{ on } \Gamma,$$

where μ and the β_i are constants with $\mu > 0$. This is an example of a stationary *convection-diffusion problem;* the Laplace term corresponds to diffusion with diffusion coefficient μ and the first order derivatives correspond to convection in the direction $\beta = (\beta_1, \beta_2)$. Let us here assume that $\mu = 1$ and that the size of $|\beta|$ is moderate (for convection-diffusion problems with $|\beta|/\mu$ large, see Chapter 9). By multiplying (2.23a) by a test function $v \in V = H_0^1(\Omega)$, integrating over Ω and using Green's formula for the Laplace-term as usual, we are led to the following variational formulation of (2.23): Find $u \in V$ such that

(2.24) $$a(u, v) = L(v) \quad \forall v \in V,$$

where

(c)

$$a(v, w) = \int_\Omega \left(\nabla v \cdot \nabla w + \left(\beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2} + v\right) w\right) dx, \quad L(v) = \int_\Omega f v \, dx.$$

It is clear that $a(\cdot, \cdot)$ is V-elliptic since if $v \in V$, we have by Green's formula:

$$\int_\Omega \left(\beta_1 \frac{\partial v}{\partial x_1} v + \beta_2 \frac{\partial v}{\partial x_2} v\right) dx = \int_\Gamma v^2 (\beta_1 n_1 + \beta_2 n_2) ds -$$

$$- \int_\Omega \left(v \, \beta_1 \frac{\partial v}{\partial x_1} + v \, \beta_2 \frac{\partial v}{\partial x_2}\right) dx = - \int_\Omega \left(\beta_1 \frac{\partial v}{\partial x_1} v + \beta_2 \frac{\partial v}{\partial x_2} v\right) dx,$$

i.e.

$$\int_\Omega \left(\beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2}\right) v \, dx = 0,$$

so that

$$a(v, v) = \int_\Omega [|\nabla v|^2 + v^2] dx = \|v\|^2_{H^1(\Omega)}.$$

Existence of a unique weak solution of (2.23) now follows from Remark 2.1. Starting from (2.24) we may formulate the following finite element method for (2.23): Find $u_h \in V_h$ such that

$$(2.25) \qquad a(u_h, v) = L(v) \qquad \forall v \in V_h,$$

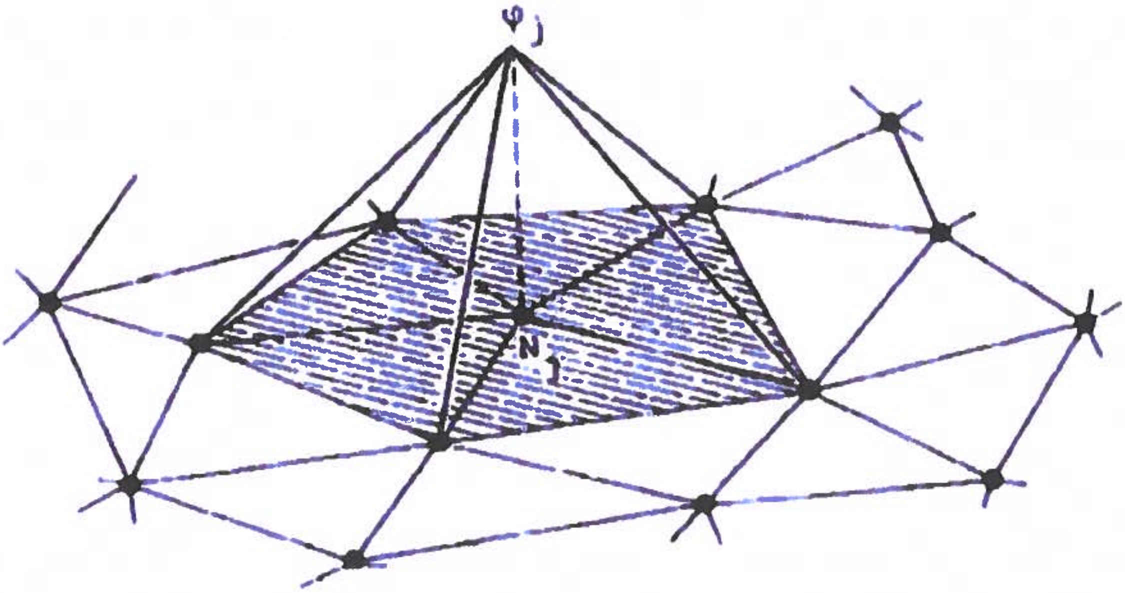
where V_h is a finite-dimensional subspace of V. If $\{\varphi_1, \ldots, \varphi_M\}$ is a basis for V_h we have as above that (2.25) is equivalent to the linear system $A\xi = b$ where $A = (a_{ij})$, $a_{ij} = a(\varphi_i, \varphi_j)$, and $b = (b_i)$, $b_i = (f, \varphi_i)$. Note that in this case the matrix A is *not* symmetric.

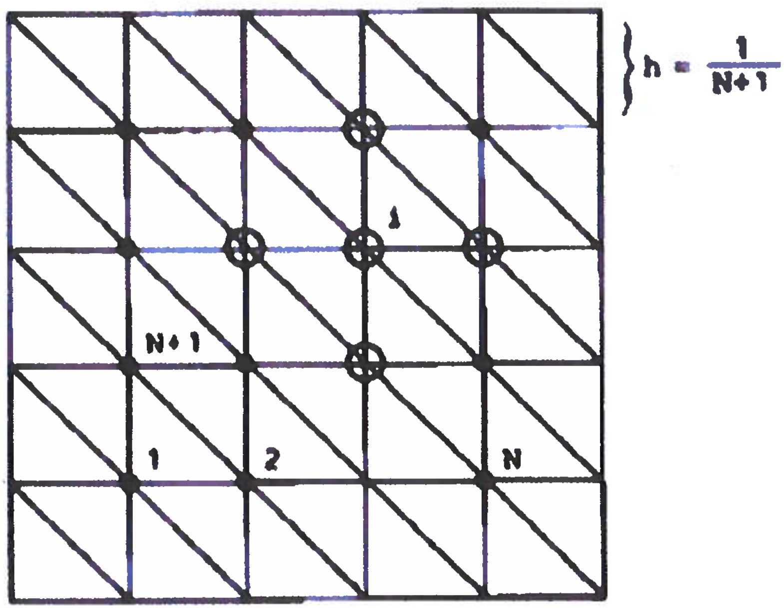
By the V-ellipticity it follows that solutions of (2.25) are unique and thus A is non-singular so that $A\xi = b$ admits a unique solution, i.e. there exists a unique solution u_h of (2.25). By the same argument as in the proof of Theorem 2.3, we also have the error estimate (here $\alpha = 1$):

$$\|u - u_h\|_{H^1(\Omega)} \leq \gamma \|u - v\|_{H^1(\Omega)} \quad \forall v \in V_h. \quad \square$$

(b)

$$\varphi_j(N_i)=\delta_{ij}\equiv\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}\qquad i,\ j=1,\ \ldots,\ M.$$





h = 1/(N+1)
i
N+1
1
2
N

(d)

The continuous problem

We shall now give an abstract formulation of the finite element method for elliptic problems of the type that we have studied in Chapter 1. This is not a goal in itself, but makes it possible to give a unified treatment of many problems in mechanics and physics so that we do not have to repeat in principle the same argument in different concrete cases. Further the abstract formulation is very easy to grasp and helps us to understand the basic structure of the finite element method.

Thus, let V be a Hilbert space with scalar product $(\cdot,\cdot)_V$ and corresponding norm $\|\cdot\|_V$ (the V-norm). Suppose that (cf Section 1.5) $a(\cdot,\cdot)$ is a bilinear form on $V \times V$ and L a linear form on V such that

(i) $a(\cdot,\cdot)$ is symmetric.

(ii) $a(\cdot,\cdot)$ is *continuous*, i.e., there is a constant $\gamma > 0$ such that

(2.1) $\quad |a(v, w)| \leq \gamma \|v\|_V \|w\|_V \qquad \forall v, w \in V,$

(iii) $a(\cdot,\cdot)$ is V-*elliptic*, i.e., there is a constant $\alpha > 0$ such that

(2.2) $\quad a(v, v) \geq \alpha \|v\|_V^2 \qquad \forall v \in V,$

(iv) L is *continuous*, i.e., there is a constant $\Lambda > 0$ such that

(2.3) $\quad |L(v)| \leq \Lambda \|v\|_V \qquad \forall v \in V.$

Let us now consider the following abstract minimization problem (M): Find $u \in V$ such that

(2.4) $\quad F(u) = \min_{v \in V} F(v),$

where

$$F(v)=\frac{1}{2}a(v,v)-L(v),$$

and consider also the following abstract variational problem (V): Find $u\in V$ such that

$$(2.5)\qquad a(u,v)=L(v)\qquad \forall v\in V.$$

Let us now first prove:

Theorem 2.1 The problems (2.4) and (2.5) are equivalent, i.e., $u\in V$ satisfies (2.4) if and only if u satisfies (2.5). Moreover, there exists a unique solution $u\in V$ of these problems and the following stability estimate holds

$$(2.6)\qquad \|u\|_V\leq\frac{\Lambda}{\alpha}.$$

Proof Existence of a solution follows from the Lax-Milgram theorem which is variant of the Riesz' representation theorem in Hilbert space theory (see e.g. [Ne], [Ci], cf also Theorem 13.1 below). The reader unfamiliar with these concepts may simply bypass this remark. To prove that (2.4) and (2.5) are equivalent, we argue exactly as in Section 1.1. We first show that if $u\in V$ satisfies (2.4), then also (2.5) holds, and we leave the proof of the reverse implication to the reader. Thus, let $v\in V$ and $\varepsilon\in R$ be arbitrary. Then $(u+\varepsilon v)\in V$ so that since u is a minimum,

$$F(u)\leq F(u+\varepsilon v)\qquad \forall\varepsilon\in R.$$

Using the notation $g(\varepsilon)=F(u+\varepsilon v)$, $\varepsilon\in R$, we thus have

$$g(0)\leq g(\varepsilon)\qquad \forall\varepsilon\in R,$$

so that g has a minimum at $\varepsilon=0$. Hence $g'(0)=0$ if the derivative $g'(\varepsilon)$ exists at $\varepsilon=0$. But

$$g(\varepsilon)=\frac{1}{2}a(u+\varepsilon v,\ u+\varepsilon v)-L(u+\varepsilon v)$$

$$=\frac{1}{2}a(u,u)+\frac{\varepsilon}{2}a(u,v)+\frac{\varepsilon}{2}a(v,u)+\frac{\varepsilon^2}{2}a(v,v)-L(u)-\varepsilon L(v)$$

$$=\frac{1}{2}a(u,u)-L(u)+\varepsilon a(u,v)-\varepsilon L(v)+\frac{\varepsilon^2}{2}a(v,v),$$

where we used the symmetry of $a(\cdot,\cdot)$. It follows that

$$0=g'(0)=a(u,v)-L(v).$$

which proves (2.5). To prove the stability result we choose $v=u$ in (2.5) and use (2.2) and (2.3) to obtain

$$\alpha\|u\|_V^2 \leq a(u, u) = L(u) \leq \Lambda\|u\|_V,$$

which proves (2.6) upon division by $\|u\|_V \neq 0$. Finally, the uniqueness follows from the stability estimate (2.6) since if u_1 and u_2 are two solutions so that $u_i \in V$ and

$$a(u_i, v) = L(v) \qquad \forall v \in V, \ i=1, 2,$$

then by subtraction we see that $u_1 - u_2 \in V$ satisfies

$$a(u_1 - u_2, v) = 0 \qquad \forall v \in V.$$

Applying the stability estimate to this situation (with $L \equiv 0$, i.e. $\Lambda = 0$) we conclude that $\|u_1 - u_2\|_V = 0$, i.e. $u_1 = u_2$. □

Remark 2.1 Even without the symmetry condition (i) and with only (ii)–(iv) satisfied, one can prove that there exists a unique $u \in V$ such that

$$a(u, v) = L(v) \qquad \forall v \in V,$$

and the stability estimate (2.6) of course holds (cf Example 2.6 below). In this case there is however no associated minimization problem. □

(e)

Theorem 2.4 Let $u \in V$ be the solution of (2.5) and $u_h \in V_h$ that of (2.9) where $V_h \subset V$. Then

$$\|u-u_h\|_V \leq \frac{\gamma}{\alpha}\|u-v\|_V \quad \forall v \in V_h.$$

Proof Since $V_h \subset V$ we have from (2.5) in particular

$$a(u,w)=L(w) \qquad \forall w \in V_h,$$

so that after subtracting (2.9),

(2.14) $$a(u-u_h,w)=0 \qquad \forall w \in V_h.$$

For an arbitrary $v \in V_h$, define $w=u_h-v$. Then $w \in V_h$, $v=u_h-w$ and by (2.2) and (2.14), we have

$$\alpha\|u-u_h\|_V^2 \leq a(u_h, u-u_h)=a(u-u_h, u-u_h)+a(u-u_h, w)$$

$$=a(u-u_h, u-u_h+w)=a(u-u_h, u-v) \leq \gamma\|u-u_h\|_V\|u-v\|_V,$$

where the last inequality follows from (2.1). Dividing by $\|u-u_h\|_V$ we obtain the desired estimate. $\square$

which proves (2.5). To prove the stability result we choose $v=u$ in (2.5) and use (2.2) and (2.3) to obtain

$$\alpha\|u\|_V^2 \leq a(u, u)=L(u)\leq\Lambda\|u\|_V,$$

which proves (2.6) upon division by $\|u\|_V\neq 0$. Finally, the uniqueness follows from the stability estimate (2.6) since if u_1 and u_2 are two solutions so that
(f) $u_i\in V$ and

$$a(u_i,v)=L(v) \qquad \forall v\in V,\ i=1,2,$$

then by subtraction we see that $u_1-u_2\in V$ satisfies

$$a(u_1-u_2,v)=0 \qquad \forall v\in V.$$

Applying the stability estimate to this situation (with $L\equiv 0$, i.e, $\Lambda=0$) we conclude that $\|u_1-u_2\|_V=0$, i.e, $u_1=u_2$. □

Remark 2.1 Even without the symmetry condition (i) and with only (ii)–(iv) satisfied, one can prove that there exists a unique $u\in V$ such that

$$a(u,v)=L(v) \qquad \forall v\in V,$$

and the stability estimate (2.6) of course holds (cf Example 2.6 below). In this case there is however no associated minimization problem. □

(g) & (h)

Now let V_h be a finite-dimensional subspace of V of dimension M. Let $\{\varphi_1, \ldots, \varphi_M\}$ be a basis for V_h, so that $\varphi_i \in V_h$ and any $v \in V_h$ has the unique representation

$$(2.7) \qquad v = \sum_{i=1}^{M} \eta_i \varphi_i, \quad \text{where } \eta_i \in R.$$

We can now formulate the following discrete analogues of the problems (*M*) and (*V*): Find $u_h \in V_h$ such that

$$(2.8) \qquad F(u_h) \leq F(v) \qquad \forall v \in V_h,$$

or equivalently: Find $u_h \in V_h$ such that

$$(2.9) \qquad a(u_h, v) = L(v) \qquad \forall v \in V_h.$$

As in Section 1.2 we see that (2.9) is equivalent to

$$a(u_h, \varphi_j)=L(\varphi_j),\ j=1,\dots, M.$$

Using the representation

$$(2.10)\qquad u_h=\sum_{i=1}^{M}\xi_i\varphi_i,\quad \xi_i\in R,$$

(2.9) can be written as

$$\sum_{i=1}^{M}a(\varphi_i,\varphi_j)\xi_i=L(\varphi_j),\ j=1,\dots, M,$$

or, in matrix form,

$$(2.11)\qquad A\xi=b,$$

where $\xi=(\xi_i)\in R^M$, $b=(b_i)\in R^M$ with $b_i=L(\varphi_i)$, and $A=(a_{ij})$ is an $M\times M$ matrix with elements $a_{ij}=a(\varphi_i,\varphi_j)$. From the representation (2.7), we have

$$a(v,v)=a(\sum_{i=1}^{M}\eta_i\varphi_i,\sum_{j=1}^{M}\eta_j\varphi_j)=\sum_{i,j=1}^{M}\eta_i a(\varphi_i,\varphi_j)\eta_j=\eta\cdot A\eta,$$

$$L(v)=L(\sum_{j=1}^{M}\eta_j\varphi_j)=\sum_{j=1}^{M}\eta_j L(\varphi_j)=b\cdot\eta,$$

where the dot denotes the usual scalar product in R^M:

$$\xi\cdot\eta=\sum_{i=1}^{M}\xi_i\eta_i.$$

It follows that (2.8) may be formulated as

$$(2.12)\qquad \frac{1}{2}\xi\cdot A\xi-b\cdot\xi=\underset{\eta\in R^M}{\mathrm{Min}}\ [\frac{1}{2}\eta\cdot A\eta-b\cdot\eta].$$

Note: for $\vec{\beta}\neq\vec{0}$ the matrix is not symmetric!

We also have, recalling (2.2),

$$\eta\cdot A\eta=a(v,v)\geq\alpha\|v\|_V^2>0,$$

if $v\neq 0$, i.e. if $\eta\neq 0$. Since also $a(\varphi_i,\varphi_j)=a(\varphi_j,\varphi_i)$, this proves the following result.

Theorem 2.2 The stiffness matrix A is symmetric and positive definite.

We can now prove the following basic result where the equivalence follows as above

$\boxed{10.2}$

Bi-harmonic model

(a)

$$\boxed{\begin{array}{l}\Delta\Delta u = f \quad , \Omega \subset \mathbb{R}^2 \\ u|_{\partial\Omega} = \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0\end{array}}$$

work out

$$\underset{\text{in } \mathbb{R}^2}{\Delta\Delta u} = \frac{\partial^4 u}{\partial x_1^4} + 2\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u}{\partial x_2^4}$$

multiply with testfunctions $v \in \mathcal{H}_0^{2,2}(\Omega)$ and integrate over Ω

$$\underbrace{\int_\Omega \Delta\Delta u \cdot v \, d\vec{x}}_{\text{Green}} = \int_\Omega f \cdot v \, d\vec{x} =: L(v)$$

$$= \int_{\partial\Omega} \cancel{\frac{\partial}{\partial n}(\Delta u) v \, ds} - \int_\Omega \nabla(\Delta u) \circ \nabla u \, d\vec{x}$$

$=0$ on $\partial\Omega$

$$= -\int_\Omega \nabla(\Delta u) \circ \nabla v \, d\vec{x}$$

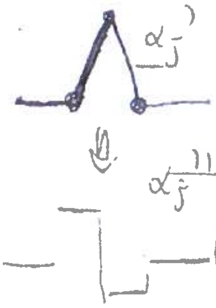
$$\underset{\text{Green}}{=} -\int_{\partial\Omega} \cancel{\Delta u \frac{\partial v}{\partial n} ds} + \int_\Omega \Delta u \Delta v \, d\vec{x} = \underbrace{\int_\Omega \Delta u \Delta v \, d\vec{x}}_{:= a(u,v)}$$

$=0$ on $\partial\Omega$

(b)

$$\text{in 1d: } \int_{\Omega_N} u'' v'' dx = \int_\Omega f v \, dx$$

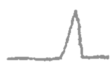
$$u \approx \sum_{j=1}^{N} U_j \alpha_j$$



α_j piecewise linear OK

α_j'' "safer to take" α_j piecewise quadratic

piecewise cubic OK



$\alpha_j'' = \sum \delta$-functions ✗

α_j: piecewise quadratic

$\alpha_j' =$ piecewise linear

$\alpha_j'' =$ piecewise constant

$\boxed{10.3}$

"intro"

Before going into the discussion of the numerical methods for (8.2) we shall briefly indicate some of the main properties of the exact solution u of (8.2). For simplicity we will then consider the following one-dimensional model problem modelling heat conduction in a bar (cf (1.3)):

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f \qquad 0<x<\pi,\ t>0, \tag{8.3a}$$

$$u(0,t)=u(\pi,t)=0 \qquad t>0, \tag{8.3b}$$

$$u(x,0)=u^0(x) \qquad 0<x<\pi. \tag{8.3c}$$

In the case $f=0$, we have by separation of variables that the solution of (8.3) is given by

$$u(x,t)=\sum_{j=1}^{\infty} u_j^0 e^{-j^2 t}\sin(jx). \tag{8.4}$$

(a)

where

$$u_j^0 = \sqrt{2/\pi} \int_0^\pi u^0(x)\sin(jx)dx, \qquad j=1, 2, \ldots,$$

are the Fourier coefficients of the initial data u^0 with respect to the orthonormal system $\{\sqrt{2/\pi}\sin(jx)\}_{j=1}^\infty$ in $L_2(0,\pi)$. By (8.4) we see that $u(x,t)$ is a linear combination of sine waves $\sin(jx)$ with frequencies j and amplitudes $u_j^0 \exp(-j^2t)$. We may say that each component $\sin(jx)$ lives on a time scale of order $O(j^{-2})$ since $\exp(-j^2t)$ is very small for j^2t moderately large. In particular we have that high frequency components quickly get damped. Thus, the solution $u(x,t)$ will become smoother as t increases. This of course fits with the intuitive idea of the nature of a diffusion process such as heat conduction. However, in general $u(x,t)$ will not be smooth for t small, and we may have that $\|\dot{u}(t)\| = \|\dot{u}(\cdot,t)\| \to \infty$ as $t \to 0$, where $\|\cdot\|$ denotes the $L_2(0,\pi)$-norm. More precisely, the size of the derivates of u (with respect to t or x) for t small will depend on how quickly the Fourier coefficients u_j^0 decay with increasing j: For example, if $u^0(x) = \pi - x$ for $0<x<\pi$, then $u_j^0 = C/j$, in which case $\|\dot{u}(t)\| \sim Ct^{-\alpha}$ with $\alpha = 3/4$ as $t \to 0$, and if $u^0(x)$ is the "hat function" $u^0(x) = \min(x, \pi - x)$ for $0<x<\pi$, then $u_j^0 = C/j^2$ in which case $\|\dot{u}(t)\| \sim Ct^{-\alpha}$ with $\alpha = 1/4$ as $t \to 0$ (cf Problem 8.1). If u_j^0 decays faster than $j^{-2.5}$ as $j \to \infty$, then $\|\dot{u}(t)\|$ will be bounded as $t \to 0$, but higher derivatives may still be unbounded. In principle, the "smoother" the initial function u^0 is, the more rapidly u_j^0 decays as $j \to \infty$. Note that here a "smooth" initial function has to satisfy in particular the boundary conditions (8.3b).

An initial phase for t small where certain derivatives of u are large, is called an *initial transient*. Thus the exact solution of a parabolic problem in general will have an initial transient where certain derivatives are large, but the solution will become smoother as t increases. This fact is of importance when solving a parabolic problem numerically, since it is advantageous to vary the mesh size (in time and space) according to the smoothness of the exact solution u and thus use a fine mesh where u is non-smooth and increase the mesh size as u becomes smoother. Note that transients may also occur for $t>0$ if for example the right hand side f (or the boundary conditions) in (8.1)–(8.3) vary abruptly in time.

The basic stability estimates in our context for the problems (8.2) and (8.3) are in the case $f=0$:

(8.5) $$\|u(t)\| \leq \|u^0\|, \qquad t \in I,$$

(8.6) $$\|\dot{u}(t)\| \leq \frac{C}{t}\|u^0\|, \qquad t \in I.$$

(b), (c) and (d)

The semi-discrete analogue of (8.2) will be based on a variational formulation of (8.2) which we now describe. Letting $V=H_0^1(\Omega)$, multiplying (8.2a) for a given t by $v\in V$, integrating over Ω and using in the usual way Green's formula, we get with the notation of Section 1.4:

$$(\dot{u}(t),v)+a(u(t),v)=(f(t),v).$$

Thus, we are led to the following variational formulation of (8.2): Find $u(t)\in V$, $t\in I$, such that

(8.7a) $$(\dot{u}(t),v)+a(u(t),v)=(f(t),v) \qquad \forall v\in V,\ t\in I,$$

(8.7b) $$u(0)=u^0.$$

Now, let V_h be a finite-dimensional subspace of V with basis $\{\varphi_1,\dots,\varphi_M\}$. For definiteness we shall assume that Ω is a polygonal convex domain and that V_h consists of piecewise linear functions on a quasi-uniform triangulation of Ω with mesh size h and satisfying the minimum angle condition (4.1). Replacing V by the finite-dimensional subspace V_h we get the following semi-discrete analogue of (8.7): Find $u_h(t)\in V_h$, $t\in I$, such that

(8.8a) $$(\dot{u}_h(t),v)+a(u_h(t),v)=(f(t),v) \qquad \forall v\in V_h,\ t\in I,$$

(8.8b) $$(u_h(0),v)=(u^0,v) \qquad \forall v\in V_h.$$

Let us rewrite (8.8) using the representation

(8.9) $$u_h(t,x)=\sum_{i=1}^{M}\xi_i(t)\varphi_i(x), \qquad t\in I,$$

with the time-dependent coefficients $\xi_i(t)\in R$. Using (8.9) and taking $v=\varphi_j$, $j=1,\dots,M$, in (8.8), we get

$$\sum_{i=1}^{M} \dot\xi_i(t)(\varphi_i,\varphi_j)+\sum_{i=1}^{M} \xi_i(t)a(\varphi_i,\varphi_j)=(f(t),\varphi_j), \quad j=1,\ldots,M, \quad t\in I,$$

$$\sum_{i=1}^{M} \xi_i(0)\,(\varphi_i,\varphi_j)=(u^0,\varphi_j) \qquad j=1,\ldots,M.$$

or in matrix form

(8.10a) $\qquad B\dot\xi(t)+A\xi(t)=F(t), \quad t\in I,$

(8.10b) $\qquad B\xi(0)=U^0,$

where $B=(b_{ij})$, $A=(a_{ij})$, $F=(F_i)$, $\xi=(\xi_i)$, $U^0=(U_i^0)$,

$$b_{ij}=(\varphi_i,\varphi_j)=\int_\Omega \varphi_i\varphi_j dx,$$

$$a_{ij}=a(\varphi_i,\varphi_j)=\int_\Omega \nabla\varphi_i\cdot\nabla\varphi_j\, dx,$$

$$F_i(t)=(f(t),\varphi_i), \; U_i^0=(u^0,\varphi_i).$$

Recall that both the mass matrix B and the stiffness matrix A are symmetric and positive definite. Further $\varkappa(B)=0(1)$ and $\varkappa(A)=0(h^{-2})$ as $h\to 0$ (see Problem 7.6). Introducing the Cholesky decomposition $B=E^TE$ and the new variable $\eta=E\xi$, the problem (8.10) takes the slightly simpler form

$$(8.11) \qquad \begin{aligned} &\dot\eta(t)+\tilde A\eta(t)=g(t), \quad t\in I,\\ &\eta(0)=\eta^0, \end{aligned}$$

where $\tilde A=E^{-T}AE^{-1}$ is a positive definite symmetric matrix with $\varkappa(\tilde A)=0(h^{-2})$, $g=E^{-T}F$, $\eta^0=E^{-T}U^0$ and $E^{-T}=(E^{-1})^T=(E^T)^{-1}$. The solution of (8.11) is given by the following formula (see any book on ordinary differential equations):

$$(8.12) \qquad \eta(t)=e^{-\tilde A t}\eta^0+\int_0^t e^{-\tilde A(t-s)}g(s)ds, \quad t\in I.$$

The problem (8.11) (and (8.10)) is an example of a stiff initial value problem, the stiffness being related to the fact that the eigenvalues of $\tilde A$ are positive and vary considerably in size corresponding to $\varkappa(\tilde A)$ being large.

Let us now return to our semi-discrete problem in the formulation (8.8). A basic stability inequality for this problem, with for simplicity $f=0$, is obtained as follows: Taking $v=u_h(t)$ in (8.8a), we get

$$(\dot u_h(t), u_h(t))+a(u_h(t), u_h(t))=0, \quad t\in I,$$

or with as above $\|\cdot\|=\|\cdot\|_{L_2(\Omega)}$,

$$\frac{1}{2}\frac{d}{dt}\|u_h(t)\|^2+a(u_h(t),\ u_h(t))=0,$$

so that recalling also (8.8b),

$$\|u_h(t)\|^2+2\int_0^t a(u_h(s),\ u_h(s))ds=\|u_h(0)\|^2\leq\|u^0\|^2,$$

and thus in particular,

(8.13) $\qquad \|u_h(t)\|\leq\|u_h(0)\|\leq\|u^0\|, \quad t\in I.$

This estimate is clearly analogous to the estimate (8.5) for the continuous problem. Note that (8.5) may also be proved in the same way as (8.13).

For the semi-discrete problem (8.8) one can prove the following almost optimal error estimate. Recall that we are assuming, for simplicity, that Ω is a convex polygonal domain and that V_h consists of piecewise linear functions on a quasi-uniform triangulation of Ω with mesh size h.

10.5

Finite Elements and the Wave Equation

The finite element procedures for the wave equation

$$u_{tt} - u_{xx} = 0, \qquad 0 \le x \le 1, \qquad 0 \le t < \infty, \tag{4-208}$$

are similar in nature to those for the parabolic and elliptic equations. We could approximate the solution of Eqn. (4-208) by undetermined coefficients

$$\bar{u}(t, x) = \sum_{j=1}^{n} U_j \phi_j(t, x),$$

or by undetermined functions,

$$\bar{u}(t, x) = \sum_{j=1}^{n} U_j(t) \phi_j(x). \tag{4-209}$$

In the sequel we use Eqn. (4-209). Upon forming the residual R in the usual way, the Galerkin method requires that

$$\int_0^1 R(t, x)\phi_i(x)\, dx = 0, \qquad i = 1, 2, \ldots, n,$$

which becomes

$$\int_0^1 (\bar{u}_{tt} - \bar{u}_{xx})\phi_i(x)\, dx = 0, \qquad i = 1, 2, \ldots, n. \tag{4-210}$$

Upon integration by parts in the second order space derivative, Eqn. (4-210) becomes

$$\int_0^1 (\bar{u}_{tt}\phi_i + \bar{u}_x \phi_{i,x})\, dx - \bar{u}_x \phi_i \Big|_0^1 = 0, \qquad i = 1, 2, \ldots, n. \tag{4-211}$$

Upon introducing the trial function, Eqn. (4-209), Eqn. (4-211) becomes

$$\sum_{j=1}^{n} \int_0^1 \left[\frac{d^2 U_j}{dt^2} \phi_j \phi_i + U_j \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} \right] dx - \bar{u}_x \phi_i \Big|_0^1 = 0, \tag{4-212}$$

which is expressible in matrix form as

$$AU'' + BU + f = 0, \qquad A = (a_{ij}), \qquad B = (b_{ij}). \tag{4-213}$$

where

$$a_{ij} = \int_0^1 \phi_j \phi_i\, dx,$$

$$b_{ij} = \int_0^1 \frac{d\phi_j}{dx} \frac{d\phi_i}{dx}\, dx$$

$$f_i = -\bar{u}_x \phi_i \Big|_0^1,$$

and

$$U = [U_1, \ldots, U_n]^T.$$

Various time approximations have been used. With the initial conditions $U(0, x) = F(x)$, $U_t(0, x) = G(x)$ known we assume $\Delta t = k$, i.e., equal time intervals are used.

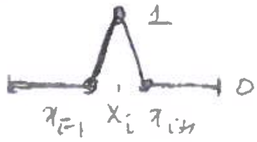
* 10.6

Least-squares approach

$$h = x_i - x_{i-1}$$

$$u_t = \varepsilon u_{xx} - \beta u_x + f \;:= \mathcal{L}(u)$$

(a) $\alpha_j(x_i) = \delta_{ij}$



$$\alpha_j(x) = \begin{cases} \frac{x_j - x_{i-1}}{h} & \text{on } [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{h} & \text{on } [x_i, x_{i+1}] \\ 0 & \text{elsewhere} \end{cases}$$

u_h:

$$\frac{\partial u_h}{\partial t} = \frac{\partial}{\partial t}\left[\sum_{j=0}^{N} u_j(t)\alpha_j(x)\right] = \sum_{j=0}^{N} \frac{\partial}{\partial t}\left(u_j(t)\alpha_j(x)\right) = \sum_{j=0}^{N} \frac{\partial u_j(t)}{\partial t} \cdot \alpha_j(x)$$

$$= \sum_{j=0}^{N} \dot{u}_j(t)\,\alpha_j(x)$$

(b) $$\left\| \frac{\partial u_h}{\partial t} - L(u_h) \right\| \neq 0$$

$$\| \dot{u}_h - L(u_h) \|^2_{L_2([0,T])} = \langle \dot{u}_h - L(u_h), \dot{u}_h - L(u_h) \rangle_{L_2([0,T])}$$

$$= \langle \dot{u}_h, \dot{u}_h \rangle_{L_2([0,T])} - 2\langle \dot{u}_h, L(u_h) \rangle_{L_2([0,T])} + \langle L(u_h), L(u_h) \rangle_{L_2([0,T])}$$

$$= \sum_{j=0}^{N}\sum_{k=0}^{N} \dot{u}_j \dot{u}_k \langle \alpha_j, \alpha_k \rangle_{L_2([0,T])} - 2\sum_{j=0}^{N} \dot{u}_j \langle \alpha_j, L(u_h) \rangle_{L_2([0,T])} + \langle L(u_h), L(u_h) \rangle_{L_2([0,T])}$$

$$\rightarrow \frac{\partial}{\partial \dot{u}_i} \| \dot{u}_h - L(u_h) \|^2_{L_2([0,T])} = 2\sum_{j=0}^{N} \dot{u}_j(t) \langle \alpha_j, \alpha_i \rangle - 2\langle \alpha_i, L(u_h) \rangle_{L_2([0,T])} \overset{\text{SET}}{=} 0$$

$$\rightsquigarrow \boxed{\sum_{j=0}^{N} \dot{u}_j(t) \langle \alpha_i, \alpha_j \rangle_{L_2([0,T])} = \langle \alpha_i, L(u_h) \rangle_{L_2([0,T])} \quad i = 0, 1, \dots, N}$$

(c)

$$\Longleftrightarrow M\dot{\underline{u}} = \varepsilon S \underline{u} - \beta C \underline{u} + \underline{F}$$

with $\underline{u} = (u_0, u_1, \dots, u_N)^T$

(M mass-matrix, S stiffness matrix)

$$\langle \alpha_i, \alpha_j \rangle_{L_2([0,1])} = \begin{cases} j=i-1: & \frac{x_i - x_{i-1}}{6} = \frac{h}{6} \\ j=i: & \frac{x_i - x_{i-1}}{3} + \frac{x_{i+1}-x_i}{3} = \frac{2}{3}h \\ j=i+1: & \frac{x_{i+1}-x_i}{6} = \frac{h}{6} \end{cases}$$

$$\langle \alpha_i, L(u_h)\rangle_{L_2} = \langle \alpha_i, \varepsilon u_{h,xx}\rangle_{L_2} - \langle \alpha_i, \beta u_{h,x}\rangle_{L_2} + \langle \alpha_i, f\rangle_{L_2}$$

integration by parts

$$= \underbrace{\text{boundary term}}_{=0} - \varepsilon \int_{x_{i-1}}^{x_{i+1}} \alpha_{i,x} \cdot u_{h,x}\, dx$$

$$= -\varepsilon \int_{x_{i-1}}^{x_i} \alpha_{i,x}\left(u_{i-1}\alpha_{i-1,x} + u_i \alpha_{i,x} + u_{i+1}\alpha_{i+1,x}\right) dx$$

$$\alpha_{i,x} = \begin{cases} 1/h & \text{on } (x_{i-1}, x_i) \\ -1/h & \text{on } (x_i, x_{i+1}) \\ 0 & \text{elsewhere} \end{cases} \qquad = \frac{\varepsilon}{h}\left[u_{i+1} - 2u_i + u_{i-1}\right]$$

$$u_{h,x} = \sum_{j=0}^{N} u_j \alpha_{j,x}$$

for example midpoint rule

$$(x_{i+1}-x_{i-1}) f(x_i, t) = 2h f(x_i, t)$$

$$\int_{x_{i-1}}^{x_{i+1}} \alpha_i f(x,t)\, dx = \ldots ? \ldots$$

numerical appro

$$= \beta\left\{u_{i-1}\int_{x_{i-1}}^{x_i} \alpha_i \alpha_{i-1,x}\, dx + u_i \int_{x_{i-1}}^{x_{i+1}} \alpha_i \alpha_{i,x}\, dx + u_{i+1}\int_{x_i}^{x_{i+1}} \alpha_i \alpha_{i+1,x}\, dx\right\}$$

$$= -\frac{\beta}{2}(u_{i+1} - u_{i-1})$$

matrix S

$$\Rightarrow \frac{h}{6}\begin{pmatrix} \ddots & & \ominus \\ & 1\ 4\ 1 & \\ \ominus & & \ddots \end{pmatrix}\begin{pmatrix} \vdots \\ \dot{u}_i \\ \vdots \end{pmatrix} = \frac{\varepsilon}{h}\begin{pmatrix} \ddots & & \ominus \\ & 1\ -2\ 1 & \\ \ominus & & \ddots \end{pmatrix}\begin{pmatrix} \vdots \\ u_i \\ \vdots \end{pmatrix} - \frac{\beta}{2}\begin{pmatrix} \ddots & & \ominus \\ & -1\ \ 0\ \ 1 & \\ \ominus & & \ddots \end{pmatrix}\begin{pmatrix} \vdots \\ u_i \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ \int_{x_{i-1}}^{x_{i+1}} \alpha_i f\, dx \\ \vdots \end{pmatrix}$$

matrix M

matrix C

vector F