$\boxed{12.1}$

$$\omega = \sqrt{1 + \alpha u_x^2}$$

$\alpha \to 0 \quad \Rightarrow \omega \to 1$

$$\Rightarrow \left[x_\xi \cdot 1 \right]_\xi = 0$$

$$\Leftrightarrow \quad x_{\xi\xi} = 0$$

$$\Rightarrow \quad x(\xi) = c_1 \xi + c_2$$

with $\begin{cases} x(0) = 0 \\ x(1) = 1 \end{cases} \Rightarrow c_1 = 1, \; c_2 = 0$

$$\Downarrow$$

$$x(\xi) = \xi$$

ξ uniform $\Rightarrow$ x uniform (in this special case)

for $u_x \approx 0$ and $\omega = u_x$ $\Big\} \Rightarrow$ discrete $\Delta x_i \, \omega_i = \text{Constant}$

$\uparrow$

$\approx 0 \quad \Rightarrow$ "$\Delta x_i \to \infty$"

(grid points move away and grid distribution becomes not useful.

12.2 $$E = \frac{1}{2}\int_0^1 \omega x_\xi^2 \, d\xi$$ "energy" (functional)

minimizing E via Euler-Lagrange equations:

$$\frac{d}{d\xi}\left(\frac{\partial \mathcal{F}}{\partial x_\xi}\right) - \frac{\partial \mathcal{F}}{\partial x} = 0 \text{ with } \mathcal{F} = \mathcal{F}(x, x_\xi) = \omega(\xi)x_\xi^2 \text{ gives:}$$

$$\frac{d}{d\xi}(2\omega x_\xi) - 0 = 0 \Leftrightarrow \boxed{\frac{d}{d\xi}[\omega \frac{dx}{d\xi}] = 0}$$

which is equivalent to the differential formulation (BV-problem)

12.4

$$\frac{\partial u}{\partial t} = \underbrace{\epsilon}_{=1} \frac{\partial^2 u}{\partial x^2} - \cancel{\beta \frac{\partial u}{\partial x}} + \cancel{s(u, x, t)}$$

Apply transformation:

$$\begin{aligned} x =& \quad x(\xi, \theta) \\ t =& \quad t(\xi, \theta) = \theta \\ \mathcal{J} :=& \quad x_\xi \end{aligned}$$

$$\Longrightarrow U_\theta - \frac{1}{\mathcal{J}} x_\theta U_\xi = \frac{\overset{=1}{\epsilon}}{\mathcal{J}} [\frac{1}{\mathcal{J}} U_\xi]_\xi - \cancel{\frac{\beta}{\mathcal{J}} U_\xi} + \cancel{s(x, U, \theta)}$$

see also next page with more details

12.4 ctd.

$$x = x(\xi, \theta) \qquad \xi \in [\xi_0, \xi_1]$$

$$t = t(\xi, \theta) = \theta \qquad \theta \in [0, T]$$

$$\underline{\underline{J}} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \theta} \\ 0 & 1 \end{pmatrix} \qquad \det \underline{\underline{J}} = \frac{\partial x}{\partial \xi}$$

$$\underline{\underline{J}}^{-1} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial \theta} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial t} \end{pmatrix} = \begin{pmatrix} 1/\frac{\partial x}{\partial \xi} & -\frac{\partial x}{\partial \theta}/\frac{\partial x}{\partial \xi} \\ 0 & 1 \end{pmatrix}$$

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t}$$

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x}$$

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial \theta} - \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}$$

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x}$$

$$u(x,t) = u(x(\xi,\theta),\theta) \stackrel{\text{def}}{=} U(\xi,\theta)$$

12.4 Ctd$_2$

Taylor: $p = x_{i+1} - x_i = H x_\xi + \frac{H^2}{2} x_{\xi\xi} + \frac{H^3}{6} x_{\xi\xi\xi} + \cdots$

$q = x_i - x_{i-1} = H x_\xi - \frac{H^2}{2} x_{\xi\xi} + \frac{H^3}{6} x_{\xi\xi\xi} + \cdots$

$\Rightarrow p - q = H^2 x_{\xi\xi} + \frac{H^4}{12} x_{\xi\xi\xi\xi} + \cdots$

$p + q = 2H x_\xi + \frac{H^3}{3} x_{\xi\xi\xi} + \cdots$

$p^2 = H^2 x_\xi^2 + H^3 x_\xi x_{\xi\xi} + \frac{H^4}{4} x_{\xi\xi}^2 + \cdots$

$q^2 = H^2 x_\xi^2 - H^3 x_\xi x_{\xi\xi} + \frac{H^4}{4} x_{\xi\xi}^2 + \cdots$

$p^3 = H^3 x_\xi^3 + O(H^4)$ $\qquad \frac{p}{q} = 1 + H \frac{x_{\xi\xi}}{x_\xi} + O(H^2)$

$q^3 = H^3 x_\xi^3 + O(H^4)$ $\qquad \frac{1}{p+q} = \frac{1}{2H x_\xi} + O(H^2)$ etc...

et cetera

Here $H = \Delta\xi$ Remember that the transformation $\xi \mapsto x$ maps the uniform grid in ξ ($\Delta\xi = H =$ constant) into the non-uniform grid in x ($\Delta x_i \neq$ constant).

Substituting the relevant expressions

Supra-convergence on a non-uniform grid with u_{xx} in the PDE can be obtained by setting $\frac{u_{xxx} x_{\xi\xi}}{3} + \frac{u_{xxxx} x_\xi^2}{12} = 0$

$\Leftrightarrow \frac{u_{xxx}^{3/4}}{3} \left(x_{\xi\xi} u_{xxx}^{1/4} + \frac{x_\xi}{4} u_{xxx}^{-3/4} u_{xxxx} x_\xi \right) = 0$

$\Leftrightarrow \frac{u_{xxx}^{3/4}}{3} \left(x_\xi u_{xxx}^{1/4} \right)_\xi = 0$

$\overset{u_{xxx} \neq 0}{\Leftrightarrow} \left(x_\xi \underbrace{u_{xxx}^{1/4}}_{w} \right)_\xi = 0$, i.e. when the monitorfunction w in the equidistribution principle equals $u_{xxx}^{1/4}$.

$\boxed{12.5}$ Given the coordinate transformation $x = x(\xi, \theta),\ t = \theta$

using the chainrule the following can be obtained

$$\begin{aligned} u_\theta &= \frac{du}{d\theta} \\ &= \frac{du}{dx}\frac{dx}{d\theta} + \frac{du}{dt}\frac{dt}{d\theta} \\ &= \frac{du}{dx}\frac{dx}{d\theta} + \frac{du}{dt} = u_x x_\theta + u_t \end{aligned}$$

Thus

$$u_t = u_\theta - u_x x_\theta \qquad (1)$$

Another use of the chainrule gives

$$\begin{aligned} u_\xi &= \frac{du}{d\xi} \\ &= \frac{du}{dx}\frac{dx}{d\xi} + \frac{du}{dt}\frac{dt}{d\xi} \\ &= \frac{du}{dx}\frac{dx}{d\xi} = u_x x_\xi \end{aligned}$$

Thus

$$u_x = \frac{u_\xi}{x_\xi} \qquad (2)$$

Using (2) with (1) gives

$$u_t = u_\theta - \frac{u_\xi}{x_\xi} x_\theta \qquad (3)$$

Given the hyperbolic PDE $u_t + c(x)u_x = f(u)$ and (2) and (3), this PDE can be transformed into $u_\theta + \beta x_\theta = \gamma u_\xi + g$.
With

$$\begin{aligned} \beta &= -\frac{u_\xi}{x_\xi}, \\ \gamma &= \cancel{=} \frac{c(x(\xi, \theta))}{x_\xi}, \\ g &= f(u). \end{aligned}$$

The Jacobian of this coordinate transformation can be given by

$$J = \begin{bmatrix} \frac{dx}{d\xi} & \frac{dx}{d\theta} \\ \frac{dt}{d\xi} & \frac{dt}{d\theta} \end{bmatrix} = \begin{bmatrix} \frac{dx}{d\xi} & \frac{dx}{d\theta} \\ 0 & 1 \end{bmatrix}.$$

The corresponding determinant can be given by

$$\mathrm{Det}(J) = \begin{vmatrix} \frac{dx}{d\xi} & \frac{dx}{d\theta} \\ 0 & 1 \end{vmatrix} = \frac{dx}{d\xi} = x_\xi.$$

The method of characteristics gives

$$\begin{aligned} \frac{dx}{ds} &= c(x), \\ \frac{dt}{ds} &= 1. \end{aligned}$$

Working with the new coordinates ξ and θ, rewriting using their dependence gives the above system too.

$\boxed{12.6\,(a)} \rightarrow$ see ex 2.9.

$\boxed{12.6\,(b)}$

Define the 'grid size ratio'

('local stretching factor'):

$$r := \frac{x_i - x_{i-1}}{x_{i+1} - x_i} := \frac{\Delta x_{i-1}}{\Delta x_i} := \frac{q}{p}$$

The truncation error T

for the central finite difference approximation

$$u_{x,i} \approx \frac{u_{i+1} - u_{i-1}}{p+q}$$

is then given by

$$-\frac{p^2 - q^2}{2(p+q)} u_{xx,i} - \frac{p^3 + q^3}{6(p+q)} u_{xxx,i} + \dots$$
$$= -\frac{1}{2} u_{xx,i}(1 - r)\Delta x_i - \frac{1}{6} u_{xxx,i}(1 - r + r^2)\Delta x_i^2 + \dots$$

$$= \frac{\Delta\xi^2}{6}\left(3x_{\xi\xi,i} u_{xx,i} + x_\xi^2 u_{xxx,i}\right) + \mathcal{O}(\Delta\xi^4)$$
$$= \Delta x_i^2\left(\frac{1}{2}\frac{x_{\xi\xi,i}}{x_{\xi,i}} u_{xx,i} + \frac{1}{6} u_{xxx,i}\right) + \mathcal{H.O.T.}$$

$\boxed{12.6\,(c)}$ follow the same procedure as in 12.4 (second part) $\Rightarrow$ $\omega = (x_\xi u_{xx}^{\frac{1}{3}})$

general transformation

12.7

2d

$$x = x(\xi, \eta, \theta)$$
$$y = y(\xi, \eta, \theta)$$
$$t = t(\xi, \eta, \theta) = \theta \quad \leftarrow \text{special choice}$$

example:

$$u_y = \frac{1}{J}\left(u_\eta x_\xi - u_\xi x_\eta\right) = \frac{u_\eta x_\xi - u_\xi x_\eta}{x_\xi y_\eta - x_\eta y_\xi}$$

$$\begin{pmatrix} dx \\ dy \\ dt \end{pmatrix} = \underline{\underline{J}} \begin{pmatrix} d\xi \\ d\eta \\ d\theta \end{pmatrix} \quad \& \quad \underline{\underline{J}} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \theta} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} d\xi \\ d\eta \\ d\theta \end{pmatrix} = \underline{\underline{J}}^{-1} \begin{pmatrix} dx \\ dy \\ d\theta \end{pmatrix} \quad \& \quad \underline{\underline{J}}^{-1} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial t} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial t} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial t} \end{pmatrix}$$

$$= \frac{1}{\det \underline{\underline{J}}} \begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial \eta}\frac{\partial x}{\partial \theta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \theta} - \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \theta} \\ 0 & 0 & 1 \end{pmatrix}$$

and
$$\boxed{\det \underline{\underline{J}} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}}$$

chain rule

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t}$$

$$\underline{\nabla} u = \frac{\partial u}{\partial \theta}\underline{\nabla}\theta + \frac{\partial u}{\partial \xi}\underline{\nabla}\xi + \frac{\partial u}{\partial \eta}\underline{\nabla}\eta$$

extra

substituting these formulas yields:

$$\frac{\partial u}{\partial x} = \frac{1}{\det \underline{\underline{J}}}\left(\frac{\partial u}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial u}{\partial \eta}\frac{\partial y}{\partial \xi}\right)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\det \underline{\underline{J}}}\left(\frac{\partial u}{\partial \eta}\frac{\partial x}{\partial \xi} - \frac{\partial u}{\partial \xi}\frac{\partial x}{\partial \eta}\right) \quad \& \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \theta} - \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} - \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial \theta} - \underline{\nabla} u \cdot \frac{\partial \underline{x}}{\partial \theta}$$

$\boxed{*}$

$\boxed{12.8\ (a)}$

By simple chain rule differentiation we have that ...

$$u(x,y) \equiv \underbrace{u(x(\xi,\eta), y(\xi,\eta))}_{u(\xi,\eta)} \quad \Rightarrow \quad \begin{aligned} u_x &= \xi_x u_\xi + \eta_x u_\eta \\ u_y &= \xi_y u_\xi + \eta_y u_\eta \end{aligned}$$

How do we evaluate terms ξ_x, η_x, ξ_y, and η_y ?

$$\begin{aligned} \xi &= \xi(x,y) \\ \eta &= \eta(x,y) \end{aligned} \qquad\qquad \begin{aligned} x &= x(\xi,\eta) \\ y &= y(\xi,\eta) \end{aligned}$$

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \xi_y \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}^{-1} = \frac{1}{J} \begin{pmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{pmatrix}$$

$$J = x_\xi y_\eta - x_\eta y_\xi$$

$$\begin{aligned} u_x &= \tfrac{1}{J}\left(y_\eta u_\xi - y_\xi u_\eta\right) \\ u_y &= \tfrac{1}{J}\left(-x_\eta u_\xi + x_\xi u_\eta\right) \end{aligned}$$

and

$$\begin{aligned} u_{xx} &= \frac{\partial}{\partial x}(u_x) = \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right) u_x \\ &= \frac{1}{J}\left(y_\eta \frac{\partial}{\partial \xi} - y_\xi \frac{\partial}{\partial \eta}\right) u_x \\ &= \ldots \\ u_{yy} &= \ldots \end{aligned}$$

Finally, $-(u_{xx} + u_{yy}) = f$, becomes

$$\boxed{\frac{-1}{J^2}\left(a\, u_{\xi\xi} - 2b\, u_{\xi\eta} + c\, u_{\eta\eta} + d\, u_\eta + e\, u_\xi\right) = f}$$

a, b, c, d, and e depend on the mapping.

$$\begin{aligned} a &= x_\eta^2 + y_\eta^2 \\ b &= x_\xi x_\eta + y_\xi y_\eta \\ c &= x_\xi^2 + y_\xi^2 \end{aligned} \qquad \begin{aligned} e &= \frac{x_\eta \beta - y_\eta \alpha}{J} \\ d &= \frac{y_\xi \alpha - x_\xi \beta}{J} \end{aligned} \qquad \begin{aligned} \alpha &= a x_{\xi\xi} - 2b x_{\xi\eta} + c x_{\eta\eta} \\ \beta &= a y_{\xi\xi} - 2b y_{\xi\eta} + c y_{\eta\eta} \end{aligned}$$

We note that in this equation all partial derivatives, including the mapping coefficients, a, b, c, d, and e, are with respect to ξ and η.

$\boxed{*}$ $\boxed{12.8(b)}$

Recall that **Laplace's equation** in $\mathbb{R}^2$ in terms of the usual (i.e., **Cartesian**) (x, y) coordinate system is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0. \tag{1}$$

The Cartesian coordinates can be represented by the polar coordinates as follows:

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \tag{2}$$

Let us first compute the partial derivatives of x, y w.r.t. ρ, φ:

$$\begin{cases} \dfrac{\partial x}{\partial \rho} = \cos\varphi, & \dfrac{\partial x}{\partial \varphi} = -\rho\sin\varphi; \\ \dfrac{\partial y}{\partial \rho} = \sin\varphi, & \dfrac{\partial y}{\partial \varphi} = \rho\cos\varphi. \end{cases} \tag{3}$$

To do so, let's compute $\dfrac{\partial u}{\partial \rho}$ first. We will use the **Chain Rule** since (x, y) are functions of (ρ, φ) as shown in (2).

$$\begin{aligned} \frac{\partial u}{\partial \rho} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho} \\ &= \frac{\partial u}{\partial x}\cos\varphi + \frac{\partial u}{\partial y}\sin\varphi \quad \text{using (3)} \\ &= \cos\varphi\frac{\partial u}{\partial x} + \sin\varphi\frac{\partial u}{\partial y}. \end{aligned} \tag{4}$$

Now, let's compute $\dfrac{\partial^2 u}{\partial \rho^2}$. Noticing that both $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are functions of (x, y)

and using (3), we have

$$
\begin{aligned}
\frac{\partial^2 u}{\partial \rho^2} &= \cos\phi \frac{\partial}{\partial \rho}\frac{\partial u}{\partial x} + \sin\phi \frac{\partial}{\partial \rho}\frac{\partial u}{\partial y} \\
&= \cos\phi\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial \rho}\right) + \sin\phi\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial \rho} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho}\right) \\
&= \cos^2\phi \frac{\partial^2 u}{\partial x^2} + 2\cos\phi\sin\phi \frac{\partial^2 u}{\partial x \partial y} + \sin^2\phi \frac{\partial^2 u}{\partial y^2}. \qquad (5)
\end{aligned}
$$

Similarly, let's compute $\frac{\partial u}{\partial \phi}$ and $\frac{\partial^2 u}{\partial \phi^2}$.

$$
\begin{aligned}
\frac{\partial u}{\partial \phi} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi} \\
&= \frac{\partial u}{\partial x}(-\rho\sin\phi) + \frac{\partial u}{\partial y}(\rho\cos\phi) \\
&= -\rho\sin\phi \frac{\partial u}{\partial x} + \rho\cos\phi \frac{\partial u}{\partial y}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial \phi^2} &= -\rho\cos\phi \frac{\partial u}{\partial x} - \rho\sin\phi \frac{\partial}{\partial \phi}\frac{\partial u}{\partial x} - \rho\sin\phi \frac{\partial u}{\partial y} + \rho\cos\phi \frac{\partial}{\partial \phi}\frac{\partial u}{\partial y} \\
&= -\rho\cos\phi \frac{\partial u}{\partial x} - \rho\sin\phi\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial \phi}\right) - \rho\sin\phi \frac{\partial u}{\partial y} + \rho\cos\phi\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial \phi} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi}\right) \\
&= -\rho\cos\phi \frac{\partial u}{\partial x} - \rho\sin\phi\left(\frac{\partial^2 u}{\partial x^2}(-\rho\sin\phi) + \frac{\partial^2 u}{\partial x \partial y}\rho\cos\phi\right) \\
&\quad - \rho\sin\phi \frac{\partial u}{\partial y} + \rho\cos\phi\left(\frac{\partial^2 u}{\partial x \partial y}(-\rho\sin\phi) + \frac{\partial^2 u}{\partial y^2}\rho\cos\phi\right) \\
&= -\rho\left(\cos\phi \frac{\partial u}{\partial x} + \sin\phi \frac{\partial u}{\partial y}\right) + \rho^2\left(\sin^2\phi \frac{\partial^2 u}{\partial x^2} - 2\cos\phi\sin\phi \frac{\partial^2 u}{\partial x \partial y} + \cos^2\phi \frac{\partial^2 u}{\partial y^2}\right)
\end{aligned}
$$

Dividing both sides by ρ^2 and using (4), we have

$$
\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} = -\frac{1}{\rho}\frac{\partial u}{\partial \rho} + \sin^2\phi \frac{\partial^2 u}{\partial x^2} - 2\cos\phi\sin\phi \frac{\partial^2 u}{\partial x \partial y} + \cos^2\phi \frac{\partial^2 u}{\partial y^2} \qquad (6)
$$

Finally, adding (5) and (6), using the obvious relation $\cos^2\phi + \sin^2\phi = 1$, we have

$$
\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} = -\frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},
$$

which can be cleaned up as:

$$
\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}.}
$$

Hence, Laplace's equation (1) becomes:

$$
\boxed{u_{xx} + u_{yy} = u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} = 0.}
$$