9.1

Theorem 1. *The Caputo fractional derivative $\mathcal{D}_C^\alpha$ with $m-1<\alpha\le m$ is consistent with the integer-order derivative $\frac{d^m}{dx^m}$ for $m \in \mathbb{N}$.*

Proof. For $g \in C^{m+1}([0,\infty])$, we write:

$$\begin{aligned}
\mathcal{D}_C^\alpha g(x) &= \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{g^{(m)}(s)}{(x-s)^{1-(m-\alpha)}} ds \\
&= \frac{1}{\Gamma(m-\alpha)} \left[-\frac{(x-s)^{m-\alpha}}{m-\alpha} g^{(m)}(s)\Big|_{s=0}^{s=x} + \int_0^x \frac{(x-s)^{m-\alpha}}{m-\alpha} g^{(m+1)}(s) ds \right] \\
&= \frac{1}{\Gamma(m-\alpha+1)} \left[0 + x^{m-\alpha} g^{(m)}(0) + \int_0^x (x-s)^{m-\alpha} g^{(m+1)}(s) ds \right]
\end{aligned}$$

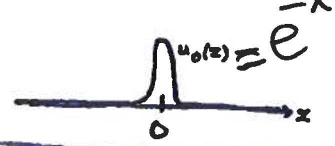
and taking the limit: $\alpha \in \mathbb{R} \to m \in \mathbb{N}$ with $m-1 \le \alpha \le m$ yields:

$$\begin{aligned}
&= \frac{1}{\Gamma(1)} \left[g^{(m)}(0) + \int_0^x g^{(m+1)}(s) ds \right] \\
&= g^{(m)}(0) + g^{(m)}(x) - g^{(m)}(0) = \frac{d^m g}{dx^m}(x).
\end{aligned}$$

! see "slide 26" lecture notes !

9.2

$$\begin{cases} \frac{\partial u}{\partial t} = D^\alpha u & , \alpha \ge 0, \ x \in]-\infty, \infty[\\ u(x,0) = u_0(x) \end{cases}$$



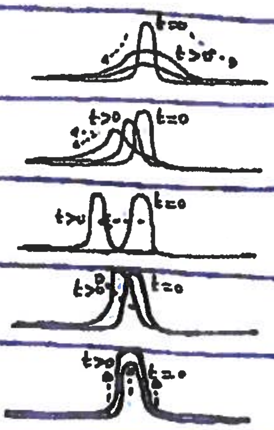
$\alpha = 2$ (heat equation)

$1 < \alpha < 2$ (fractional)

$\alpha = 1$ (transport equation)

$0 < \alpha < 1$ (fractional)

$\alpha = 0$ (ODE)



$\int_{-\infty}^{x}$; in the case we use the fractional derivative defined by $\int_x^\infty$ we see similar effects, but now "motion" to the right (instead of left as in the picture)

- symmetric fractional diffusion : take the "average" of $\int_{-\infty}^x$ and $\int_x^\infty$
- difference with "ordinary" diffusion: slower and "flatter tails" than $u_t = u_{xx}$

9.3

(a) make use of lecture notes: formula 1) in "A mysterious contradiction". (take $\alpha = 3/2$ and $k = 5$)

(b) see matlab-file on the webpage.

Suppose $h = x/k$, k is a positive integer. Using a second order difference approximation, we get

$$
\begin{aligned}
& {}_0D_x^\alpha u(x,t) \\
&= \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{1}{(x-\xi)^{\alpha-1}} \frac{\partial^2 u(\xi,t)}{\partial \xi^2}\, d\xi \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} \int_{jh}^{(j+1)h} z^{1-\alpha} \frac{\partial^2 u(x-z,t)}{\partial z^2}\, dz \\
&\approx \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} \frac{u(x-(j-1)h,t) - 2u(x-jh,t) + u(x-(j+1)h,t)}{h^2} \\
&\quad \times \int_{jh}^{(j+1)h} z^{1-\alpha}\, dz \\
&= \frac{h^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{k-1} [u(x-(j-1)h,t) - 2u(x-jh,t) + u(x-(j+1)h,t)] \\
&\quad \times [(j+1)^{2-\alpha} - j^{2-\alpha}].
\end{aligned}
$$

9.4

9.5

(a) check directly, using the definitions.

(b) apply D_C^α (or D_{RL}^α) to a linear combination of functions

(c) substitute and use "cos"-property.

(d) plot Gamma-function and also interpolating functions on top of this graph (making use of cosines ...)

see next page

(e) Another important property, which we often use in this thesis, is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. This property helps us to calculate fractional order derivatives and integrals easier. Some other Gamma function values are [6]:

$$
\begin{cases}
\Gamma(-2) = \infty \\
\Gamma(-\frac{3}{2}) = \frac{4}{3}\sqrt{\pi} \approx 2.363271 \\
\Gamma(-1) = \infty \\
\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi} \approx 0.886226 \\
\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi} \approx 1.329340 \\
\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi} \approx 3.323350 \quad \textit{et cetera.}
\end{cases}
\tag{1.15}
$$

The Gamma function $\Gamma(z)$ is defined:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt, \tag{1.7}$$

which is known to converge in the complex plane for $Re(z) > 0$ (see Figure 1.1).

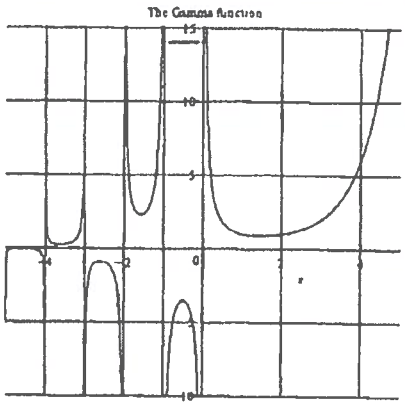


FIGURE 1.1: The Gamma function along part of the real axis.

1.2.2 Some properties of the Gamma function

One of the most important properties of the Gamma function is:

$$\Gamma(z+1) = z\Gamma(z), \tag{1.8}$$

which can be easily checked as follows:

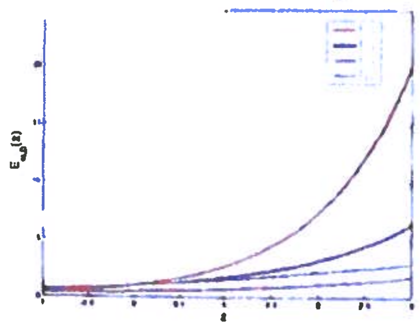
$$\Gamma(z+1) = \int_0^{\infty} e^{-t} t^{z}\, dt = [-e^{-t} t^{z}]_{t=0}^{t=\infty} + z\int_0^{\infty} e^{-t} t^{z-1}\, dt = z\Gamma(z). \tag{1.9}$$

It is obvious that $\Gamma(1) = 1$. We can conclude from (1.8):

$$\begin{cases} \Gamma(2) = 1\,\Gamma(1) = 1 = 1!, \\ \Gamma(3) = 2\,\Gamma(2) = 2\,1! = 2!, \\ \Gamma(4) = 3\,\Gamma(3) = 3\,2! = 3!, \\ \dots \\ \Gamma(n+1) = n\,\Gamma(n) = n\,(n-1)! = n!. \end{cases} \tag{1.10}$$

(f)

The Mittag-Leffler function (1.18) for different α and β values. We obtain the exponential function for $\alpha = 1$ and $\beta = 1$ ($E_{1,1}$).



.nother important formula in fractional calculus is **the Mittag-Leffler function** [26]. The Mittag-effler function shows its importantance in physics, biology, engineering and applied sciences nd it has a direct involvement in problems. The Mittag-Leffler function is a fundamental so-ition of fractional differential equations and fractional order integral equations. The most no-iceable general property of the Mittag-Leffler function is dealing with Laplace transform and symptotic expansions of these functions. The Mittag-Leffler function has two definitions in the terature:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in C,\ Re(\alpha) > 0, \tag{1.17}$$

and

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in C,\ Re(\alpha) > 0,\ Re(\beta) > 0. \tag{1.18}$$

9.6

! Check the article by Meerschaert & Tadjeran on the webpage of the course !
and the lecture notes.

9.7

(a) $\begin{cases} u'(t) = \lambda\, u(t) \,, & \lambda \in \mathbb{C} \\ u(0) = u_0 \end{cases}$

following Axelsson & Verwer (Egbv1 on webpage):

$$\begin{cases} u_0 = u_0 \\ u_{n+1} - u_{n-1} - 2\lambda \Delta t\, u_n = 0 & , n = 1, 2, \dots, N-1 \\ u_N - u_{N-1} - \lambda \Delta t\, u_N = 0 \end{cases}$$

$\vec{\eta} \overset{def}{=} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \Longrightarrow$

$$\boxed{B(\lambda \Delta t)\, \vec{\eta} = \vec{b}}$$

$$B(\lambda\Delta t) \overset{def}{=} \begin{pmatrix} -2\lambda\Delta t & 1 & & & \Theta \\ -1 & -2\lambda\Delta t & 1 & & \\ & -1 & \ddots & \ddots & \\ & & -1 & -2\lambda\Delta t & 1 \\ \Theta & & & -1 & 1-\lambda\Delta t \end{pmatrix} \quad \& \quad \vec{b} \overset{def}{=} \begin{pmatrix} u_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(b) $\begin{cases} \vec{u}_0 = \vec{u}_0 \\ \vec{u}_{n+1} - \vec{u}_{n-1} - 2A\Delta t\, \vec{u}_n = \vec{0} & , n = 1, 2, \dots, N-1 \\ \vec{u}_N - \vec{u}_{N-1} - \Delta t A \vec{u}_N = \vec{0} \end{cases}$

$$\vec{\eta} = \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_N \end{pmatrix} \Rightarrow C\, \vec{\eta} = \vec{b},$$

where $\vec{b} = \begin{pmatrix} \vec{u}_0 \\ \vec{0} \\ \vec{0} \\ \vdots \\ \vec{0} \end{pmatrix}$, $C = \begin{pmatrix} -2A\Delta t\, I & & & \Theta \\ -I & -2A\Delta t\, I & & \\ & \ddots & \ddots & \\ \Theta & & -I & -2A\Delta t\, I \\ & & -I & I - A\Delta t \end{pmatrix}$

(c)

below, read for $f(u(t))$: $S(u)$
(in exercise)

$$\frac{u_2-1}{2\Delta t} - S'(u_1) = 0 \Rightarrow \underbrace{u_2 - 2\Delta t\, S'(u_1)}_{f_1(\vec{u}) = f_1(u_1,u_2)} = \underset{b_1}{1}$$

$$\frac{u_3-u_1}{2\Delta t} - S'(u_2) = 0 \Rightarrow \underbrace{u_3 - 2\Delta t\, S'(u_2) - u_1}_{f_2(\vec{u}) = f_2(u_1,u_2,u_3)} = \underset{b_2}{0}$$

$$\underbrace{u_4 - 2\Delta t\, S'(u_3) - u_2}_{f_3(u_2,u_3,u_4)} = 0$$

$$\underbrace{u_5 - 2\Delta t\, S'(u_4) - u_3}_{f_4(u_3,u_4,u_5)} = 0$$

$\vdots$

$$\frac{u_N - u_{N-2}}{2\Delta t} - S'(u_{N-1}) = 0 \longrightarrow \underbrace{u_N - 2\Delta t\, S'(u_{N-1}) - u_{N-2}}_{f_{N-1}(u_{N-2},u_{N-1},u_N)} = \underset{b_{N-1}}{0}$$

$$\underbrace{u_N - \Delta t\, S'(u_N) - u_{N-1}}_{f_N(u_{N-1},u_N)} = \underset{b_N}{0}$$

$\Rightarrow \vec{F}(\vec{u}) = \vec{b}$ and apply, for instance, Newton-Raphson to find numerically $\vec{u}$.

$\Leftrightarrow \underbrace{\vec{F}(\vec{u}) - \vec{b}}_{= \vec{G}(\vec{u})} = \vec{0}$

nonlinear system

(or use "fsolve" in Matlab)

9.8

see lecture notes!

9.9

doubling $\Rightarrow$ $u_{tt} = -u_{xx}$ "backward wave equation"

splitting $\Rightarrow$ $\begin{cases} u_t = v \\ v_t = -u_{xx} \end{cases}$

semi-discrete: $\begin{cases} \dot{\vec{u}} = \vec{v} \\ \dot{\vec{v}} = D_2 \vec{u} \end{cases}$

$$\Rightarrow \quad \dot{\vec{\eta}} = \begin{pmatrix} 0 & I \\ D_2 & 0 \end{pmatrix} \vec{\eta}$$

check eigenvalues of $\begin{pmatrix} 0 & I \\ D_2 & 0 \end{pmatrix}$ and conclude: need BVM

9.10

doubling $\Rightarrow$ $u_{tttt} = u_{xxxxx}$

$u_1 \stackrel{d}{=} u$
$u_2 \stackrel{d}{=} u_{1,t} = u_t$
$u_3 \stackrel{d}{=} u_{2,t} = u_{tt}$
etcetera

splitting $\Rightarrow$ $\begin{cases} u_{1,t} = u_2 \\ u_{2,t} = u_3 \\ u_{3,t} = u_4 \\ u_{4,t} = u_5 \\ u_{5,t} = u_{1,xxxxx} \end{cases}$ $\xrightarrow{\text{semi-discrete}}$ $\begin{cases} \dot{\vec{u}}_1 = \vec{u}_2 \\ \dot{\vec{u}}_2 = \vec{u}_3 \\ \dot{\vec{u}}_3 = \vec{u}_4 \\ \dot{\vec{u}}_4 = \vec{u}_5 \\ \dot{\vec{u}}_5 = D_5 \vec{u}_1 \end{cases}$

matrix: $\begin{pmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ D_5 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\downarrow$

check eigenvalues

$\downarrow$

BVM