Q1

$(h \overset{d}{=} \Delta x)$

(a)

$$u_{j+2} = u_j + (2h)u_j' + \frac{(2h)^2}{2}u_j'' + \frac{(2h)^3}{6}u_j''' + \frac{(2h)^4}{24}u_j'''' + \frac{(2h)^5}{120}u_j''''' + \dots$$

$$u_{j+1} = u_j + h u_j' + \frac{h^2}{2}u_j'' + \frac{h^3}{6}u_j''' + \frac{h^4}{24}u_j'''' + \frac{h^5}{120}u_j''''' + \dots$$

$$u_j = u_j + 0 \cdot u_j' + 0 \cdot u_j'' + \dots$$

$$u_{j-1} = u_j - h u_j' + \frac{h^2}{2}u_j'' - \frac{h^3}{6}u_j''' + \frac{h^4}{24}u_j'''' - \frac{h^5}{120}u_j''''' \dots$$

$$u_{j-2} = u_j - (2h)u_j' + \frac{(2h)^2}{2}u_j'' - \frac{(2h)^3}{6}u_j''' + \frac{(2h)^4}{24}u_j'''' - \frac{(2h)^5}{120}u_j''''' \dots$$

$$\frac{-Au_{j+2} + Bu_{j+1} - Bu_{j-1} + Au_{j-2}}{h}$$

$$= \frac{1}{h}\Big\{ \cancel{-Au_j} - 2hAu_j' - \cancel{2h^2Au_j''} - \frac{8}{6}Ah^3u_j''' - \cancel{\frac{16}{24}h^4Au_j''''} - \frac{32}{120}h^5Au_j''''' + \dots$$

$$\cancel{+Bu_j} + Bhu_j' + \cancel{B\frac{h^2}{2}u_j''} + \frac{Bh^3}{6}u_j''' + \cancel{\frac{Bh^4}{24}u_j''''} + \frac{Bh^5}{120}u_j''''' \dots$$

$$\cancel{-Bu_j} + Bhu_j' - \cancel{B\frac{h^2}{2}u_j''} + B\frac{h^3}{6}u_j''' - \cancel{B\frac{h^4}{24}u_j''''} + B\frac{h^5}{120}u_j''''' \dots$$

$$\cancel{+Au_j} - 2hAu_j' + \cancel{2h^2Au_j''} - \frac{8}{6}Ah^3u_j''' + \cancel{\frac{16}{24}Ah^4u_j''''} - \frac{32}{120}h^5Au_j''''' \dots \Big\}$$

$$= \frac{1}{h}\Big\{ hu_j'(-4A+2B) + h^3u_j'''\left(-\frac{16}{6}A + \frac{2}{6}B\right) + h^5u_j'''''\left(\frac{-64}{120}A + \frac{2}{120}B\right) + \dots \Big\}$$

$$\begin{cases} -4A+2B = 1 \\ -\frac{16}{6}A + \frac{2}{6}B = 0 \end{cases} \Leftrightarrow \begin{cases} -2A+B = \frac{1}{2} \\ -8A+B = 0 \end{cases} \Leftrightarrow \begin{cases} B = \frac{1}{2}+2A \\ -8A+B=0 \end{cases} \Leftrightarrow \begin{array}{l} -8A+2A+\frac{1}{2}=0 \\ -6A = -\frac{1}{2} \\ A = \frac{1}{12} \\ B = \frac{8}{12} \end{array}$$

$$\frac{-\frac{1}{12}u_{j+2} + \frac{8}{12}u_{j+1} - \frac{8}{12}u_{j-1} + \frac{1}{12}u_{j-2}}{h} \Longleftarrow$$

$$\Rightarrow \boxed{\frac{-u_{j+2}+8u_{j+1}-8u_{j-1}+u_{j-2}}{12\,\Delta x}}$$

(b) error term: $\left(\frac{-64}{120}A+\frac{2}{120}B\right)h^5 u_j^{(5)}/h$

$$=\left(\frac{-64}{120}\cdot\frac{1}{12}+\frac{2}{120}\,\frac{8}{12}\right)h^4 u_j^{(5)}$$

$$=\frac{1}{12\cdot 120}(-64+16)\,h^4 u_j^{(5)}$$

$$=\frac{1}{12\cdot 120}\cdot -48\,h^4 u_j^{(5)}$$

$$=-\frac{1}{30}h^4 u_j^{(5)}$$

$$\Rightarrow C_4=(\pm)\frac{1}{30}\,,\quad p=5$$

$\boxed{Q2}$ $\dot{y} = f(t,y)$ (*)

(a) Insert exact solution y into the approximation

$$y^{n+1} = y^n + \Delta t \left[\frac{1}{7} f(t^{n+1}, y^{n+1}) + \frac{6}{7} f(t^n, y^n)\right]$$

$$\Rightarrow y(t^{n+1}) \approx y(t^n) + \Delta t \Big(\frac{1}{7} f(t^{n+1}, y(t^{n+1})) + \frac{6}{7} f(t^n, y(t^n))\Big]$$

Re-arrange and use (*):

$$y(t^{n+1}) - y(t^n) - \Delta t \left[\frac{1}{7} f(t^{n+1}, y(t^{n+1})) + \frac{6}{7} f(t^n, y(t^n))\right]$$

$$= y(t^{n+1}) - y(t^n) - \Delta t \left[\frac{1}{7} \dot{y}(t^{n+1}) + \frac{6}{7} \dot{y}(t^n)\right]$$

$$\Rightarrow y(t^{n+1}) - y(t^n) - \Delta t \left[\frac{1}{7} \dot{y}(t^{n+1}) + \frac{6}{7} \dot{y}(t^n)\right]$$

$$= \left[\cancel{y(t^n)} + \cancel{\Delta t \dot{y}(t^n)} + \frac{1}{2}(\Delta t)^2 \ddot{y}(t^n) + \frac{1}{6}(\Delta t)^3 \dddot{y}(t^n) + O((\Delta t)^4)\right]$$
$$- \cancel{y(t^n)} - \Delta t \left[\frac{1}{7}\Big(\cancel{\dot{y}(t^n)} + \Delta t \ddot{y}(t^n) + \frac{1}{2}(\Delta t)^2 \dddot{y}(t^n) + O((\Delta t)^3)\Big) + \cancel{\frac{6}{7}\dot{y}(t^n)}\right]$$

$$= \left(\frac{1}{2} - \frac{1}{7}\right)(\Delta t)^2 \ddot{y}(t^n) + O((\Delta t)^3),$$

$$\overset{(\pm)}{=} \frac{5}{14} (\Delta t)^2 \ddot{y}(t^n) + O((\Delta t)^3)$$

local truncation error ($\Rightarrow$ first-order method, global error $O(\Delta t)$)

b) $\dot{y} = \lambda y \quad , \quad z = \lambda \Delta t$

$$y^{n+1} = y^n + \Delta t \left[\frac{1}{7} \lambda y^{n+1} + \frac{6}{7} \lambda y^n \right]$$

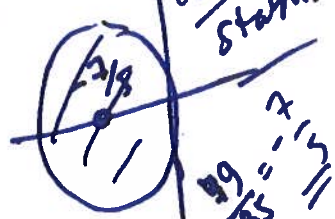
$$= y^n + \frac{z}{7} y^{n+1} + \frac{6z}{7} y^n$$

$$\left(1 - \frac{1}{7} z\right) y^{n+1} = \left(1 + \frac{6}{7} z\right) y^n$$

$$y^{n+1} = \underbrace{\frac{1 + \frac{6}{7} z}{1 - \frac{1}{7} z}}_{= R(z)} y^n$$

$R(z)$: $\quad y^{n+1} = y^n + \Delta t\left(\frac{1}{7}\lambda y^{n+1} + \frac{6}{7}\lambda y^n\right)$

(c) $\quad \left(1 - \frac{\Delta t}{7}\lambda\right) y^{n+1} = \left(1 + \frac{6\Delta t}{7}\lambda\right) y^n$



cond. stable.

$\frac{49}{-35} = -\frac{7}{5}$

$$y^{n+1} = \boxed{\frac{1 + \frac{6}{7}z}{1 - \frac{1}{7}z}}\, y^n \qquad R(z)$$

circled centered at $\left(-\frac{90}{70}, 0\right)$

radius $\left(-\frac{7}{5}, 0\right)$

$\frac{90}{35}$

$|R(z)| = 1 \qquad \left|1 + \frac{6}{7}z\right| = \left|1 - \frac{1}{7}z\right|$

$z = x + iy \qquad \left|\left(1 + \frac{6}{7}x\right) + \left(\frac{6}{7}y\right)i\right| = \left|\left(1 - \frac{1}{7}x\right) + \left(-\frac{1}{7}y\right)i\right|$

$$\left(1 + \frac{6}{7}x\right)^2 + \left(\frac{6}{7}y\right)^2 = \left(1 - \frac{1}{7}x\right)^2 + \left(\frac{1}{7}y\right)^2$$

$$\cancel{1} + \frac{12}{7}x + \frac{36}{49}x^2 + \frac{36}{49}y^2 = \cancel{1} - \frac{2}{7}x + \frac{1}{49}x^2 + \frac{1}{49}y^2$$

$\frac{35}{49}x^2 + \frac{14 \cdot 7}{49}x + \frac{35}{49}y^2 = 0 \qquad 35x^2 + 98x + 35y^2 = 0$

$x^2 + \frac{98}{35}x + y^2 = 0 \quad \left(x + \frac{98}{70}\right)^2 + y^2 = \left(\frac{98}{35}\right)^2$

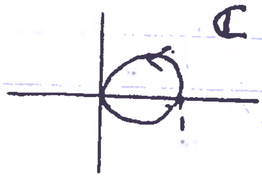
Q3

We consider the ODE: $\dot{y}(t) = f(y(t))$

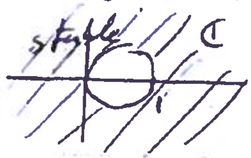
a) The boundary locus is the curve that describes the boundary of the stability region of the time integration scheme. The "eigenvalues" of your function $f(y(t))$ must fall inside the stability region in order to have a stable numerical solution.

b) The function $g(\phi) : [0, 2\pi) \Rightarrow \mathbb{C}$ describe the boundary locus.

F1) $g_1(\phi) = 1 - e^{-i\phi}$



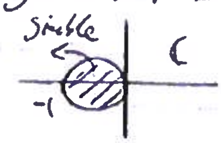
belongs to M_3 Euler Backward



c)

F6) $g_6(\phi) = e^{i\phi} - 1$

belongs to M_1 Euler Forward



F3) $g_3(\phi) = i\sin\phi$

belongs to M_4 Explicit Midpoint

Now F2) $g_2(\phi) = 3/2 - 2e^{-i\phi} + 1/2\, e^{-2i\phi}$ is an extension of Euler Backward $g_1(\phi)$, but not as complicated as F5) $g_5(\phi) = 1 + e^{-i\phi} + \frac{1}{2}(1-e^{-i\phi})^2 + \frac{1}{3}(1-e^{-i\phi})^3$

F2) is expanded up to 2nd order while F5) up to third order.

Hence F_2) belongs to M_2) (BDF_2 : expansion of EB)
and F_5) belongs to M_6) (BDF_3 : a tougher/complicated expansion of EB)

Lastly remains F_4) connected to M_5)

↳ F_4) has region in the imaginary axis or outside the imaginary axis, but it is more complicated than F_3) which belongs to the simple midpoint method.

So
$F_1 - M_3$
$F_2 - M_2$
$F_3 - M_4$
$F_4 - M_5$
$F_5 - M_6$
$F_6 - M_1$

Q3

(d) $y^{n+3} = y^{n+2} + \frac{\Delta t}{12}\left[5f(y^n) - 16f(y^{n+1}) + 23f(y^{n+2})\right]$

$\pi(z; \zeta)$

$\underline{\underline{r=3}}$

$$\underbrace{0}_{\alpha_0} \cdot y^n + \underbrace{0}_{\alpha_1} \cdot y^{n+1} \underbrace{-1}_{\alpha_2} y^{n+2} + \underbrace{1}_{\alpha_3} \cdot y^{n+3}$$

$$= \Delta t \left\{ \underbrace{\frac{5}{12}}_{\beta_0} f(y^n) - \underbrace{\frac{16}{12}}_{\beta_1} f(y^{n+1}) + \underbrace{\frac{23}{12}}_{\beta_2} f(y^{n+2}) + \underbrace{0}_{\beta_3} \cdot f(y^{n+3}) \right\}$$

$\Rightarrow$ explicit

$$\Leftrightarrow \sum_{j=0}^{3} \alpha_j y^{n+j} = \Delta t \sum_{j=0}^{r} \beta_j f(y^{n+j})$$

$$\rho(\zeta) = \sum_{j=0}^{3} \alpha_j \zeta^j, \quad \sigma(\zeta) = \sum_{j=0}^{3} \beta_j \zeta^j$$

$= 0 \cdot \zeta^0 + 0 \cdot \zeta^1 - 1 \cdot \zeta^2 + 1 \cdot \zeta^3 =$

$= \zeta^3 - \zeta^2 = \zeta^2(\zeta - 1)$

$= \frac{5}{12}\zeta^0 - \frac{16}{12}\zeta^1 + \frac{23}{12}\zeta^2 + 0 \cdot \zeta^3$

$= \frac{5}{12} - \frac{16}{12}\zeta + \frac{23}{12}\zeta^2$

stab. pol. $\pi(\zeta;z) = \rho(\zeta) - z\,\sigma(\zeta)$

$$= \zeta^3 - \zeta^2 - z\left(\tfrac{5}{12} - \tfrac{16}{12}\zeta + \tfrac{23}{12}\zeta^2\right)$$

$$= \zeta^3 - \left(1 + \tfrac{23}{12}z\right)\zeta^2 + \tfrac{16}{12}z\zeta - \tfrac{5}{12}z$$

(d) $3y^{n+2} - 4y^{n+1} + y^{n} = \Delta t \cdot 2f(y^n)$

$\underline{r=2}$

$\alpha_0 = 1,\ \alpha_1 = -4,\ \alpha_2 = 3$

$\rho(\zeta) = 1 - 4\zeta + 3\zeta^2$

$\beta_0 = 2,\ \beta_1 = 0,\ \beta_2 = 0$ $\qquad \rho'(\zeta) = -4 + 6\zeta$

$\sigma(\zeta) = 2$

$\rho(1) = 1-4+3 = 0$ ✓ $\qquad \Longrightarrow$ consistent

$\rho'(1) = -4+6 = 2 = \sigma(1)$ ✓

roots of $\rho(\zeta)$: $3\zeta^2 - 4\zeta + 1 = 0$

$$\zeta_{1,2} = \frac{4 \pm \sqrt{16-12}}{6}$$

$$= \frac{4}{6} \pm \frac{1}{6}\sqrt{4} = \frac{4}{6} \pm \frac{2}{6} = \frac{6}{6} \text{ or } \frac{2}{6}$$

$$= 1 \text{ or } \frac{1}{3}$$

simple

$\underline{\text{zero-stable}} \Longleftarrow$

$$y^{n+2} - 4y^{n+1} + 3y^n = -\Delta t \cdot -2f(y^n)$$

$q=2$ $\quad \alpha_0 = 3, \ \alpha_1 = -4, \ \alpha_2 = 1$

$\beta_0 = -2, \ \beta_1 = 0, \ \beta_2 = 0 \rightarrow \sigma(\zeta) = 2$

$\rho(\zeta) = 3 - 4\zeta + \zeta^2$

$\rho(1) = 3 - 4 + 1 = 0$ ✓

$\rho'(\zeta) = -4 + 2\zeta$

$\rho'(1) = -4 + 2 = -2$

~~ρ~~ consistent

zero-stable?

$\zeta^2 - 4\zeta + 3 = 0$

$\zeta_{1,2} = \dfrac{4 \pm \sqrt{16 - 12}}{2} = 2 \pm \frac{1}{2}\sqrt{4}$

$= 2 \pm 1$

$= 3$ of 1

$3 > 1$

$\not\leq$ root condition

NO

Q4

(a) $u(x,t) = e^{-(x+t)^2}$

(b) Von Neumann : "FTFS" ; $\sigma \stackrel{def}{=} \gamma \frac{\Delta t}{\Delta x}$

$\gamma = 1$: $\sigma = \frac{\Delta t}{\Delta x}$, $\gamma = -1$: $\sigma = -\frac{\Delta t}{\Delta x}$

$$\cancel{e^{ik_m x}} e^{a(t+\Delta t)} = \left(1+\frac{\Delta t}{\Delta x}\right)\cancel{e^{ik_m x}} e^{at} - \frac{\Delta t}{\Delta x} e^{ik_m(x+\Delta x)} \cancel{e^{at}}$$

$$g = e^{a\Delta t} = \cdots = \left(1+\frac{\Delta t}{\Delta x}\right) - \left(\frac{\Delta t}{\Delta x}\right)\left(\cos(k_m\Delta x) + i\sin(k_m\Delta x)\right)$$

$$|g| = \cdots = 1 + \underbrace{2\sigma(1+\sigma)}_{>0}\underbrace{(1-\cos(k_m\Delta x))}_{\geq 0} > 1$$

$\Rightarrow$ UNSTABLE

$$|g| = 1 + \underbrace{4\sigma}_{<0}(1+\sigma)\underbrace{\sin^2(\ldots)}_{>0} < 1 \quad \text{if} \quad -1 < \underbrace{-\frac{\Delta t}{\Delta x}}_{=\sigma} < 0$$

$\Updownarrow$

$\frac{\Delta t}{\Delta x} < 1$ conditionally stable

(c) for $\frac{\gamma\Delta t}{\Delta x} = -1 \Rightarrow (\Delta x)^2 = \gamma^2(\Delta t)^2$ (*)

$$\tau = \frac{u_t\Delta t + u_{tt}\frac{(\Delta t)^2}{2} + \ldots}{\Delta t} + \gamma\frac{u_x\Delta x + u_{xx}\frac{(\Delta x)^2}{2} + \ldots}{\Delta x}$$

$$= u_t + \gamma u_x + \frac{u_{tt}}{2}\Delta t + \frac{u_{ttt}(\Delta t)^2}{6} + \ldots + \frac{\gamma}{2}u_{xx}\Delta x + \frac{\gamma}{6}u_{xxx}(\Delta x)^2 + \ldots$$

make use of $u_{tt} = \gamma^2 u_{xx}$, $u_{ttt} = -\gamma^3 u_{xxx}$ etcetera

and (*) $\Rightarrow$ at least second order accurate

5

$u_t - u_{xx} = f(t)$

multiply equation by test function v ($v|_{\partial\Omega} = 0$) and integrate over Ω

$$\int_\Omega u_t v \, d\Omega - \int_\Omega u_{xx} v \, d\Omega = \int_\Omega f(t) v \, d\Omega$$

integration by part:

$$\int_\Omega u_t v \, d\Omega - \int_\Omega [u_x v]_x \, d\Omega + \int_\Omega u_x v_x \, d\Omega = \int_\Omega f(t) v \, d\Omega$$

use Green's formula:

$$\int_\Omega u_t v \, d\Omega - \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds + \int_\Omega u_x v_x \, d\Omega = \int_\Omega f v \, d\Omega$$

$$\Rightarrow \int_\Omega u_t v \, d\Omega + \int_\Omega u_x v_x \, d\Omega = \int_\Omega f v \, d\Omega$$

choose $u = \sum u_i^{(t)} \varphi_i$, $v = \varphi_j$ then:

$$\sum \dot{u}_i \int_\Omega \varphi_i \varphi_j \, d\Omega + \sum u_i \int \varphi_i' \varphi_j' \, d\Omega = \int f \varphi_j \, d\Omega \quad , \quad \varphi' = \frac{\partial \varphi}{\partial x}$$

$$\Rightarrow \begin{cases} A \dot{\vec{u}} + B \vec{u} = \vec{f} \\ \vec{u} = \sum u_i \varphi_i \quad , \quad \vec{f} = \int_\Omega f \varphi_i \, d\Omega \end{cases}$$

$$A = \begin{pmatrix} \ddots & & 0 \\ & \langle \varphi_i, \varphi_j \rangle & \\ 0 & & \ddots \end{pmatrix} \quad , \quad B = \begin{pmatrix} \ddots & & 0 \\ & \langle \varphi_i', \varphi_j' \rangle & \\ 0 & & \ddots \end{pmatrix} \quad , \quad f = \begin{pmatrix} \vdots \\ \langle f, \varphi_j \rangle \\ \vdots \end{pmatrix}$$

$$\langle \varphi_i, \varphi_j \rangle = \int_\Omega \varphi_i \varphi_j \, d\Omega \quad ; \quad \langle \varphi_j, \varphi_j \rangle = \int_\Omega \varphi_j^2 \, d\Omega$$

$$\langle \varphi_i', \varphi_j' \rangle = \int_\Omega \varphi_i' \varphi_j' \, d\Omega \quad ; \quad \langle \varphi_j', \varphi_j' \rangle = \int_\Omega \varphi_j'^2 \, d\Omega$$

$$\langle f, \varphi_j \rangle = \int_\Omega f \varphi_j \, d\Omega$$

- A mass-matrix , tridiagonal in 1D , symetric positive definite.

B stiffness-matrix , tridiagonal in 1D , symetric positive definite

Q6

(a) $w = 1 \Rightarrow (X_\xi)_\xi = 0$

$x(0) = 0,\ x(1) = 1$

$$\Rightarrow \left.\begin{aligned} x(\xi) &= A\xi + B \\ x(0) &= B = 0 \\ x(1) &= A \end{aligned}\right\} x(\xi) = \xi$$

uniform

$\Rightarrow X$ also uniform

(b) $g = X_\xi = \dfrac{dx}{d\xi} > 0$

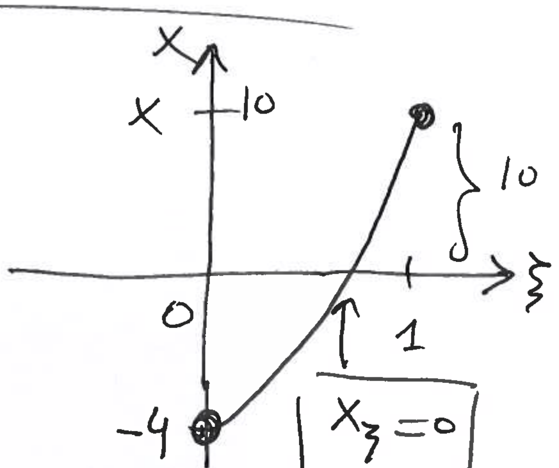
(c) $x(\xi) = 4\xi^3 + \xi^2 - 4\xi$

$X_\xi = 12\xi^2 + 2\xi - 4$

$X_{\xi\xi} = 24\xi + 2$

$X_{\xi\xi} > 0$ for $\xi > 0$

$\Rightarrow$ "$\smile$"

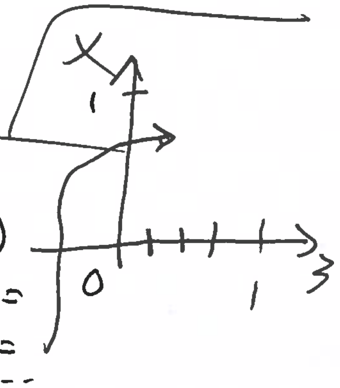


$\boxed{X_\xi = 0}$ for this ξ-value

(d) $x(\xi) = \xi^4 \Rightarrow X_\xi = 4\xi^3 > 0$ for $\xi \in [0,1]$

(e) $X_i = X(\xi_i) = \cdots = \begin{cases} \text{(c)}\ 4\xi_i^3 + \xi_i^2 - 4\xi \\ \text{(d)}\ \xi_i^4 \end{cases} \Rightarrow$ $X_1 =$, $X_2 =$, ---

$\xi_1 = \cdots,\ \xi_2 = \cdots$ etc



Q7 a)

$\dot{y}(t) = -y(t)$, $y(0) = 1$

Substitute e^{rt}, $r \in \mathbb{R}$

$\Rightarrow re^{rt} = -e^{rt} \Rightarrow r = -1$

So $y(t) = C \cdot e^{-rt}$ solves equation $C \in \mathbb{R}$

$y(0) = 1 \Rightarrow C = 1 \Rightarrow y(t) = e^{-rt}$

Let $y^n = e^{-rt}$, then $y^{n+1} = e^{-r(t+\Delta t)}$ so

$$\frac{y^{n+1} - y^n}{1 - e^{-\Delta t}} = \frac{e^{-(t+\Delta t)} - e^{-t}}{1 - e^{-\Delta t}} = e^{-t}\,\frac{e^{-\Delta t} - 1}{1 - e^{-\Delta t}} = -e^{-t}\,\frac{1 - e^{-\Delta t}}{1 - e^{-\Delta t}}$$

$= -e^{-t} = -y^n$, so indeed the scheme is an exact FD scheme for the ODE, as $y^0 = 1$ and $e^{-0} = 1$

Q7 b)

(b1) $\phi(\Delta t) = 1 - e^{-\Delta t}$

(b2) If one expands ϕ we find $\phi(\Delta t) = \Delta t$ + H.O.T.
with H.O.T. (higher order terms) in Δt (like $(\Delta t)^2$ for example)

(b3) $\phi(\Delta t) = \Delta t$

(b4) no it is local, we would discritize $y(t)$ by y^n in the local setting on the RHS => Which is done.

If the scheme would have y^{n+1} or y^{n-1} on right hand side (for example)
it would have been non local

$\boxed{Q8}$

$$u_t = u^q u_x - \varepsilon u_{xxx}, \quad q \geq 2, \quad \varepsilon > 0$$

$$\mathcal{H} = \int_{-\infty}^{\infty} \left[\frac{\varepsilon}{2} u_x^2 + \frac{u^{q+2}}{(q+1)(q+2)} \right] dx$$

(a) $$\frac{\delta \mathcal{H}}{\delta u} = \frac{\partial}{\partial u}\left\{ \frac{\varepsilon}{2} u_x^2 + \frac{u^{q+2}}{(q+1)(q+2)} \right\} - \frac{d}{dx}\left\{ \frac{\partial}{\partial u_x}\left[\frac{\varepsilon}{2} u_x^2 + \frac{u^{q+2}}{(q+1)(q+2)} \right]\right\}$$

$$= \frac{u^{q+1}}{q+1} - \frac{d}{dx}\left\{ \varepsilon u_x \right\} = \frac{u^{q+1}}{q+1} - \varepsilon u_{xx}$$

(b) $$u_t = \mathcal{J} \frac{\delta \mathcal{H}}{\delta u} = \frac{\partial}{\partial x}\left\{ \frac{u^{q+1}}{q+1} - \varepsilon u_{xx} \right\}$$

$\mathcal{J} = \frac{\partial}{\partial x}$
(skew-symm.)

$$= \frac{q+1}{q+1} u^q u_x - \varepsilon u_{xxx}$$

$$= u^q u_x - \varepsilon u_{xxx}$$

(c) check lecture notes on Hamiltonian PDEs

Q10

(a) $c \neq 0$: hyperbolic PDE

$c = 0$: elliptic PDE

(b) $\begin{cases} u_t = v \\ v_t = u_{tt} = -\alpha v - \beta u + c^2 u_{xx} \end{cases} \rightarrow \begin{pmatrix} \dot{\vec{u}} \\ \dot{\vec{v}} \end{pmatrix} = \begin{pmatrix} 0 & I \\ \boxed{c^2 D_{2c} - \beta I} & \boxed{-\alpha I} \end{pmatrix} \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}$

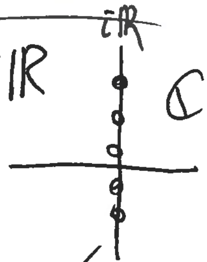
(c) $\alpha = \beta = 0$, $c = 1$: $\lambda^2 = \mu \in \mathbb{R}^- \Rightarrow \lambda \in i\mathbb{R}$

~~EF~~

EB ʃ

~~MP~~ ʃ

~~BVM~~



(d) $\alpha = 2$, $\beta = 0$, $c = 2$

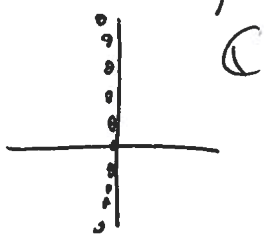
$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 + 4\mu}}{2} = -1 \pm \sqrt{1+\mu} \approx -1 \pm \sqrt{\mu}$$

EF ?...

EB ʃ

~~MP~~

BVM ʃ



(e) $\alpha = c = 0$, $\beta = -100$

$\lambda^2 = 100$, $\lambda = \pm 10$

~~EF~~, ~~EB~~

~~MP~~, ~~BVM ʃ~~