Numerical Methods for Time-Dependent PDEs

Test (tentamen): WISL602

June 12, 2019: 10:00-13:00

Question 1 (10 points)

Classify the PDE: $\alpha u_{tt} + \beta u_t + \gamma u_{xx} - x^2 u_x = \sin(x)\mathrm{e}^{-t}$
with $\alpha \geq 0,\ \beta \in \mathbb{R},\ \gamma \neq 0$. (hyperbolic, parabolic, elliptic?)

Question 2 (15 points)

Consider the ODE: $\dot{y}(t) = f(y(t))$.

a) Find the *stability* polynomial $\pi(z;\zeta)$ for the *Leapfrog* method:

$$\frac{y^{n+1} - y^{n-1}}{2\Delta t} = f(y^n).$$

b) Find the *stability* function[1] $R(z)$ for the 3/8-*rule* *RK*-method (Kutta, 1901):

$$\left\{ \begin{array}{l} k_1 = \Delta t\ f(y^n), \\ k_2 = \Delta t\ f(y^n + \frac{1}{3}k_1), \\ k_3 = \Delta t\ f(y^n - \frac{1}{3}k_1 + k_2, \\ k_4 = \Delta t\ f(y^n + k_1 - k_2 + k_3), \\ y^{n+1} = y^n + \frac{1}{8}[k_1 + 3k_2 + 3k_3 + k_4]. \end{array} \right\}$$

c) What does *zero-stability* mean and why is this an important concept? What is the role of $R(z)$ in terms of *stability regions* and stability properties of time-integration methods?

Question 3 (15 points)

Consider the *advection* equation: $u_t + c\ u_x = 0,\ c \in \mathbb{R}$.

a) Show, using *Von Neumann stability analysis*, that the forward (in space and time) scheme

$$u_j^{n+1} = (1 + \frac{c\Delta t}{\Delta x})u_j^n - \frac{c\Delta t}{\Delta x}u_{j+1}^n$$

is always unstable for $c > 0$.

b) Work out the *local truncation error* of this scheme and show that the method is first-order accurate. What happens in the special case $\frac{c\Delta t}{\Delta x} = -1$ and why?[2]

[1] It is convenient to define $z = \lambda\Delta t$.

[2] Check this for the second-order term only, but predict what could happen for higher-order terms in the error.

Question 4 (25 points)

Consider the PDE:

$$\begin{cases} u_t = \alpha\ u_{xxxx}, \quad \alpha \in \mathbb{R}, \quad x \in [0,1], \quad t > 0, \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = 0,\ u(x,0) = u_0(x). \end{cases}$$

a) Use the *method of undetermined coefficients* to find a *second-order* accurate approximation of u_{xxxx} at the grid point $x_j = j\Delta x,\ j = 0, 1, ..., J$ with $\Delta x = \frac{1}{J}$ of the form (determine the constants A and B):

$$\frac{Au_{j+2} + Bu_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}}{(\Delta x)^4}.$$

b) Apply the first step in the (vertical) *Method-Of-Lines* (M.O.L.) and give the resulting system of ODEs in terms of the matrix $\mathcal{D}_{4c}$ (representing u_{xxxx}) and the solution vector $\vec{u}$. Discuss how you include the four *boundary conditions* in the matrix $\mathcal{D}_{4c}$.

c) For the second step in M.O.L. we could use: i) Euler-Forward (EF), ii) Euler-Backward (EB) or iii) the Trapezoidal method (TM). Sketch the *computational stencil* of the full space-time finite-difference method for the three cases.

d) Which *stability criterion* of the form $-\alpha\frac{(\Delta t)^p}{(\Delta x)^q} \leq C$ do you expect to hold for the method EF as time-integrator in the case $\alpha < 0$? (specify the values of p and q, no calculations needed here)

e) It is given that the eigenvalues of the $\mathcal{D}_{2c}$-matrix (for u_{xx} at x_j) are:

$$\lambda_j = \frac{2}{(\Delta x)^2}(\cos(j\pi\Delta x) - 1), \quad j = 1, ..., J-1.$$

Discuss the effect of the value of $\alpha \in \mathbb{R}$ on the eigenvalues distribution of $\alpha\mathcal{D}_{4c}$ in the complex plane[3], and on the numerical stability of EF, EB and TM.

Question 5 (10 points)

We apply a *finite element method* with *piecewise linear functions*, both as approximating and test functions, to discretize the damped wave equation:

$$u_{tt} + \beta u_t - \Delta u = 0, \quad (x,y) \in [0,1] \times [0,1] \subset \mathbb{R}^2$$

with initial conditions $u(x,y,0) = u_0(x,y), u_t(x,y,0) = v_0(x,y)$ and homogeneous Dirichlet boundary conditions.

[3] You may use the fact that the eigenvectors of $\mathcal{D}_{2c}$ and $\mathcal{D}_{4c}$ are identical.

Give the resulting system of time-dependent ODEs in terms of matrices $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and solution vector $\vec{u}$, after the first step in the M.O.L..

Bonus (**5 points**): Describe the *structure* of the matrices $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and, in particular, their *non-zero* elements.

Question 6 (10 points)

a) Check that the *Leapfrog* method

$$\frac{y^{n+1} - y^{n-1}}{2\Delta t} = \sqrt{y^n}, \quad y^0 = 1, \; y^1 = \frac{1}{4}(\Delta t)^2 + \Delta t + 1$$

is an *exact* finite difference (FD) scheme for: $\dot{y}(t) = \sqrt{y(t)}$ with $y(0) = 1$.

b) Give two important ingredients of a *nonstandard* FD scheme, when compared to a standard FD scheme.

Question 7 (15 points)

Consider the θ-method for the ODE $\dot{y} = f(t, y)$:

$$y^{n+1} = y^n + \Delta t[\theta f(t^{n+1}, y^{n+1}) + (1 - \theta) f(t^n, y^n)]$$

with parameter $\theta \in [0, 1]$.

a) Which values of θ yield, respectively, Euler-Forward, Euler-Backward and the Trapezoidal method?

b) Give the *order of approximation* of the θ-method (check the dependance on the value of θ).

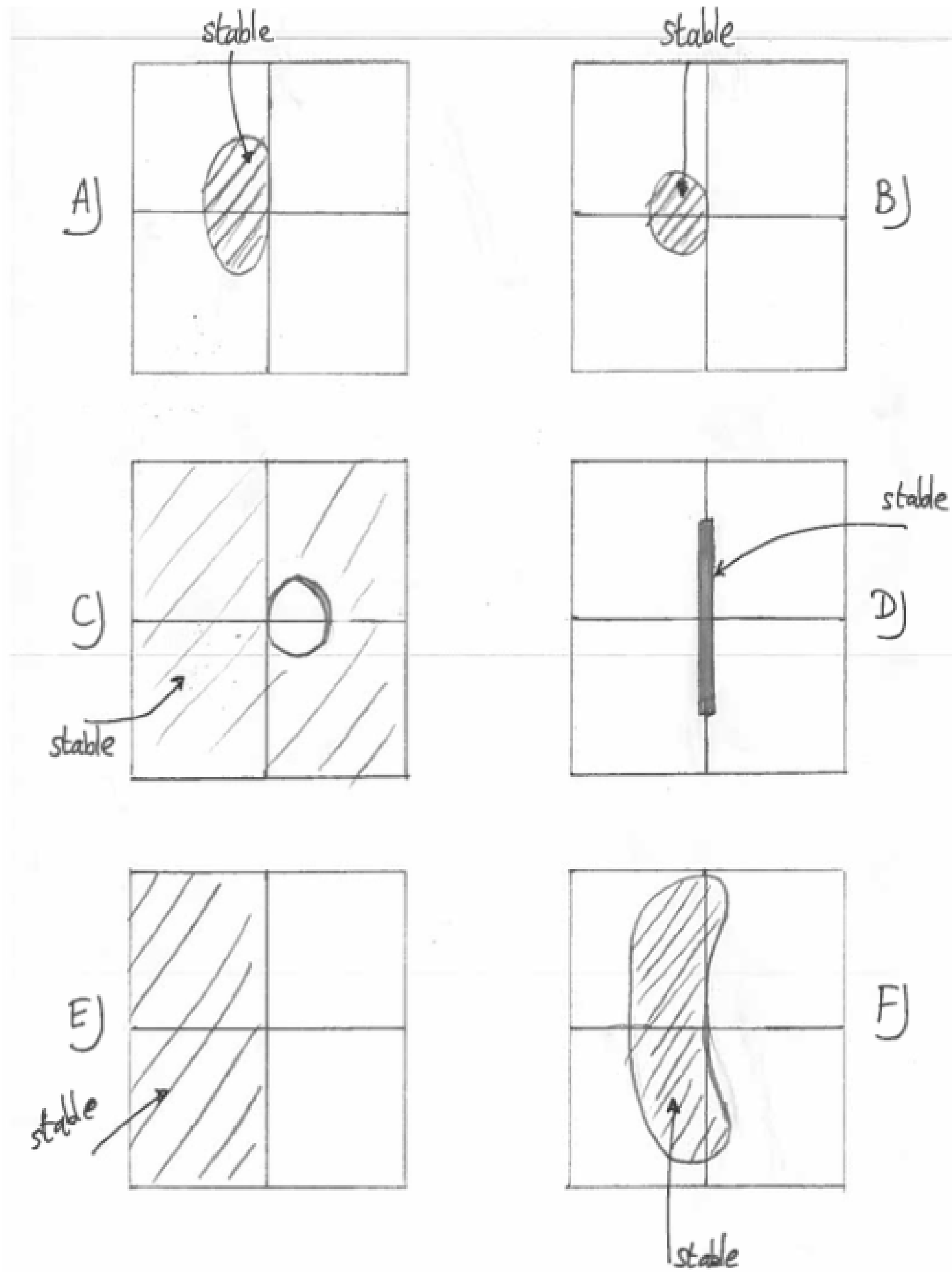
c) For which values of θ, do we obtain an *explicit*, and for which θ, do we obtain an *implicit* method?

d) Give the stability function $R_\theta(z)$ of the θ-method.

e) Consider also two other stability functions:

$$R_2(z) = 1 + z + \frac{1}{2}z^2 \quad \text{and} \quad R_3(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4.$$

On the *next page* you find plots of six stability regions A), B), ..., F). Connect five of these plots to a stability function from the set $\{ R_\theta, R_2, R_3 \}$ and specify θ, if needed. Connect each plot to one time-integration method. The sixth, remaining, plot: which method belongs to this stability region? (no stability function should be given for this case)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