Q1 $\boxed{\text{max 10 pt}}$

$$\underbrace{\alpha}_{\geq 0} u_{tt} + \underbrace{\beta}_{\in \mathbb{R}} u_t + \underbrace{\gamma}_{\neq 0} u_{xx} - x^2 u_x = \sin(x) e^{-t}$$

$\alpha, \gamma > 0$ } elliptic, $\forall \beta$

$\alpha > 0, \gamma < 0$ } hypabolic, $\forall \beta$

$$\left.\alpha = 0, \begin{cases} \beta > 0, \gamma < 0 \\ \beta < 0, \gamma > 0 \end{cases}\right\} \text{parabolic}$$

max 15 pt

Q2

max 5 pt

a) $\dfrac{y^{n+1} - y^{n-1}}{2\Delta t} = f(y^n)$ $\quad \sum_{j=0}^{r} \alpha_j \zeta^j$

Σ

$\Leftrightarrow \quad y^{n+1} = y^{n-1} + 2\Delta t \underbrace{f(y^n)}_{\text{here } = \lambda y^n} = \underbrace{2\Delta t \lambda}_{= z} y^n + y^{n-1}$

$\Rightarrow \quad \rho(\zeta) = \zeta^2$

$\sigma(\zeta) = -2z\zeta - 1$

$= \rho(\zeta) - z\sigma(\zeta)$

$\Rightarrow \quad \boxed{\pi(\zeta; z) = \zeta^2 - 2z\zeta - 1}$

max 5 pt

c)

* zero-stable $\Leftrightarrow$ method satisfies the root condition

$\Leftrightarrow$ roots of $\rho(\zeta)$ are all inside unit circle in $\mathbb{C}$

(those on the circle : simple roots)

convergence $\Leftrightarrow$ consistent and zero-stable

* $|R(z)| \leq 1 \Rightarrow$ num. stability ("A-stability")

$z = \lambda \Delta t \in \mathbb{C}$

a region in $\mathbb{C}$

(stab. region)

Q2 $\quad z = \lambda \Delta t$, apply method to $y' = \lambda y$

b)

max 5pt

$$\Rightarrow k_1 = z y^n$$

$$k_2 = z\left(y^n + \tfrac{1}{3} z y^n\right) = \left(z + \tfrac{1}{3} z^2\right) y^n$$

$$k_3 = z\left(y^n - \tfrac{1}{3} z y^n + \left(z + \tfrac{1}{3} z^2\right)\right) y^n$$

$$= \left(z - \tfrac{1}{3} z^2 + z^2 + \tfrac{1}{3} z^3\right) y^n$$

$$= \left(z + \tfrac{2}{3} z^2 + \tfrac{1}{3} z^3\right) y^n$$

$$k_4 = z\left(y^n + z y^n - \left(z + \tfrac{1}{3} z^2\right) y^n + \left(z + \tfrac{2}{3} z^2 + \tfrac{1}{3} z^3\right) y^n\right)$$

$$= z y^n + \cancel{z^2 y^n} - \cancel{z^2 y^n} - \tfrac{1}{3} z^3 y^n + z^2 y^n + \tfrac{2}{3} z^3 y^n + \tfrac{1}{3} z^4 y^n$$

$$= y^n\left(z + z^2 + \tfrac{1}{3} z^3 + \tfrac{1}{3} z^4\right)$$

$$y^{n+1} = y^n + \tfrac{1}{8}\Big[z y^n + 3\left(z + \tfrac{1}{3} z^2\right) y^n + 3\left(z + \tfrac{2}{3} z^2 + \tfrac{1}{3} z^3\right) y^n + \left(z + z^2 + \tfrac{1}{3} z^3 + \tfrac{1}{3} z^4\right) y^n\Big]$$

$$= y^n + \tfrac{1}{8}\left[8z + 4z^2 + \tfrac{4}{3} z^3 + \tfrac{1}{3} z^4\right] y^n$$

$$= \underbrace{\left[1 + z + \tfrac{1}{2} z^2 + \tfrac{1}{6} z^3 + \tfrac{1}{24} z^4\right]}_{= R(z)} y^n$$

Q3

max 15pt

$$u_t + c u_x = 0, \quad c \in \mathbb{R}$$

max 8pt

a) FTFS: $u_j^{n+1} = \Big(1 + \underbrace{\frac{c\Delta t}{\Delta x}}_{=\sigma}\Big) u_j^n - \frac{c\Delta t}{\Delta x} u_{j+1}^n$

$c>0$

Von Neumann $\Rightarrow$ $e^{ik_m x} e^{a(t+\Delta t)} = (1+\sigma) e^{ik_m x} e^{at} - \sigma e^{ik_m (x+\Delta x)} \cdot e^{at}$

Divide by $e^{ik_m x} e^{at}$ $\Rightarrow$

amplification factor: $\underbrace{e^{a\Delta t}}_{=g} = (1+\sigma) - \sigma e^{ik_m \Delta x}$

$= 1+\sigma - \sigma\cos(k_m\Delta x) - \sigma i \sin(k_m \Delta x)$

imaginary part

take absolute value squared: $|g|^2 = [\underbrace{1+\sigma-\sigma\cos(k_m\Delta x)}_{\text{real part}}]^2 + \sigma^2 \sin^2(k_m\Delta x)$

$= (1+\sigma)^2 - 2\sigma(1+\sigma)\cdot\cos(k_m\Delta x) + \sigma^2\cos^2(k_m\Delta x) + \sigma^2 \sin^2(k_m\Delta x)$ $\quad \sigma^2$

$1+2\sigma+\sigma^2 = 2\sigma(1+\sigma)$

$= 1 + \underbrace{2\sigma(1+\sigma)}_{>0}\underbrace{(1-\cos(k_m\Delta x))}_{\geq 0} > 1$

$\Rightarrow$ unstable

---

not part of the question

[$c<0$ $\quad |g|^2 = 1 + \underbrace{\underbrace{4\sigma}_{<0}(1+\sigma)\underbrace{\sin^2\left(\frac{k_m\Delta x}{2}\right)}_{>0}}_{<1 \text{ for } |\sigma| \text{ sufficiently small}}$]

max 5pt

b) $\frac{c\Delta t}{\Delta x} = -1$ & re-write formula: $\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$

local truncation error:

$$\Rightarrow \frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_j^n}{\Delta x}$$

using Taylor

$$= \frac{u_t \Delta t + u_{tt}\frac{(\Delta t)^2}{2} + \dots}{\Delta t} + c\,\frac{u_x \Delta x + u_{xx}\frac{(\Delta x)^2}{2} + \dots}{\Delta x}$$

$$= \underbrace{u_t + cu_x}_{\substack{=0 \\ (\text{PDE})}} + \underbrace{\frac{u_{tt}}{2}(\Delta t)^1 + \frac{c}{2}(\Delta x)^1 u_{xx}}_{O(\Delta t) + O(\Delta x)} + O((\Delta t)^2) + O((\Delta x)^2)$$

(⇒ first order in time and space)

if $c\frac{\Delta t}{\Delta x} = -1$, then $u_{tt} = (u_t)_t$

$$= (-cu_x)_t = (-cu_t)_x$$

$$= c^2 u_{xx}$$

and $\frac{\Delta t}{2}u_{tt} + \frac{c\Delta x}{2}u_{xx} = 0$

$\frac{\Delta t}{2}c^2 u_{xx} + \frac{c\Delta x}{2}u_{xx}$

$= \left(\frac{\Delta t}{2}c^2 + \frac{c\Delta x}{2}\right)u_{xx}$

$= 0$ if $\frac{c\Delta t}{\Delta x} = -1$

⇒ higher-order error

following this general rule it is easy to see:

Actually, all terms vanish (using mathematical induction you can prove this)

→ not part of the question

! ⇒ error $= 0$

Q4

max 15 pt

$$u_t = \alpha\, u_{xxxx}$$

max 5 pt a)

$$A u_{j+2} = A u_j + 2Ahu' + 2Ah^2u'' + \frac{4}{3}Ah^3u''' + \frac{2}{3}Ah^4u'''' + \dots$$

$$B u_{j+1} = B u_j + Bhu' + \frac{B}{2}h^2u'' + \frac{B}{6}h^3u''' + \frac{B}{24}h^4u'''' + \dots$$

$$6u_j = 6u_j + 0 + 0 + \dots$$

$$-4u_{j-1} = -4u_j + 4hu' - 2h^2u'' + \frac{1}{3}h^3u''' - \frac{1}{6}h^4u'''' \dots$$

$$u_{j-2} = u_j - 2hu' + 2h^2u'' - \frac{4}{3}h^3u''' + \frac{2}{3}h^4u'''' + \dots \quad /+$$

$$A+B+6-4+1 = 0 \Rightarrow \boxed{A+B=-3}$$

$$2A+B+4-2=0 \Rightarrow \boxed{2A+B=-2}$$

$$\boxed{A=1,\ B=-4}$$

$$2A+\frac{B}{2}-2+2=0 \Rightarrow 2A+\frac{B}{2}=0$$

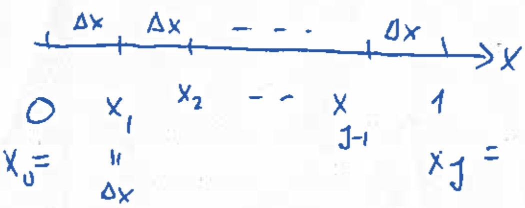
$$\frac{4}{3}A+\frac{1}{6}B+\frac{2}{3}-\frac{4}{3}=0 \Rightarrow \frac{4}{3}A+\frac{1}{6}B=\frac{2}{3}$$

$$\frac{2}{3}A+\frac{1}{24}B-\frac{1}{6}+\frac{2}{3}=1 \Rightarrow \frac{2}{3}A+\frac{1}{24}B=\frac{1}{2}$$

max 5pt b)

$$\begin{cases} \dot{\vec{u}} = \alpha \cdot D_{4c}\vec{u} \\ \vec{u}(0)=\vec{u}_0 \end{cases} \qquad \rightarrow j=2,\dots,J-2$$

$$\frac{u_{j+2}-4u_{j+1}+6u_j-4u_{j-1}+u_{j-2}}{h^2}$$



$u(0,t)=0: u_0=0$

$u_x(0,t)=0: u_1=u_0=0$

$u(1,t)=0: u_J=0$

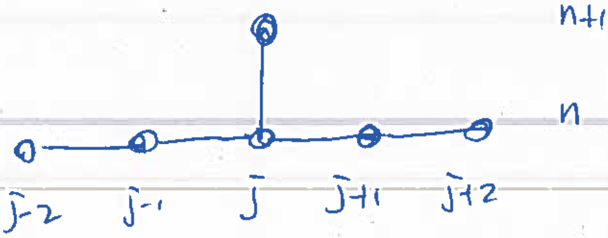
$u_x(1,t)=0: u_{J-1}=0$

for $j=2,3$

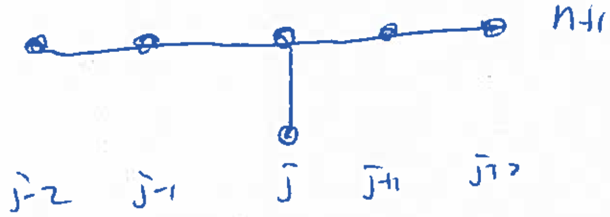
for $j=J-3, J-2$

max 5pt

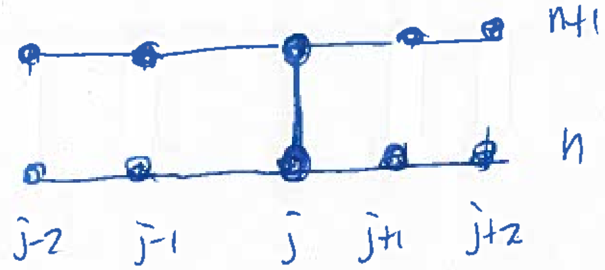
c) EF with b)



EB with b)



TM with b)



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max 5pt

d) $-\alpha \dfrac{(\Delta t)^p}{(\Delta x)^q} \le C$: EF $\begin{cases} p = 1 \\ q = 4 \end{cases}$

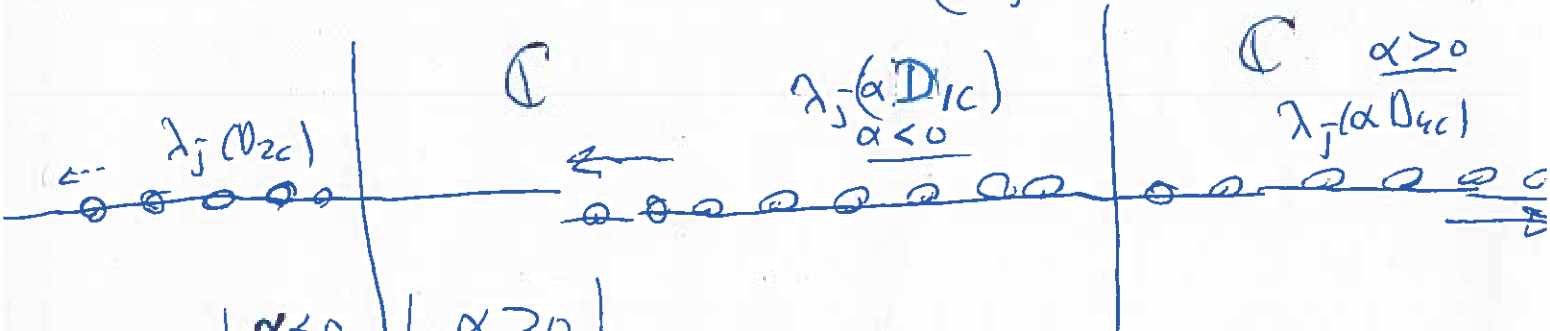
$\alpha < 0$

---

max 5pt

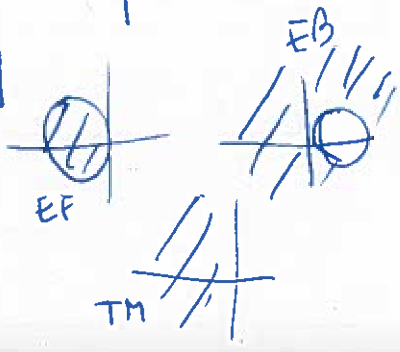
e) $\lambda_j(D_{2c}) = \dfrac{2}{(\Delta x)^2}\left(\cos(j\pi\Delta x) - 1\right) \quad j = 1, \dots, J-1$

$$\lambda_j(\alpha D_{4c}) = \alpha \cdot \lambda_j\left((D_{2c})^2\right) = \frac{2\alpha}{(\Delta x)^4}\left(\cos(j\pi\Delta x) - 1\right)^2$$



	$\alpha < 0$	$\alpha > 0$
EF	cond. stab.	uncond. unst.
EB	uncond stab	cond stab
TM	uncond stab	uncond. unst

check stability regions



max 10pt

Q5

$$u_{tt} + \beta u_t - \Delta u = 0$$

multiply testfunction v $u_{tt} v + \beta u_t v - v \Delta u = 0$

integrate $\Rightarrow \int_\Omega u_{tt} v \, d\Omega + \beta \int_\Omega u_t v \, d\Omega - \int_\Omega v \Delta u = 0$

"green" $\Rightarrow \int_\Omega u_{tt} v \, d\Omega + \beta \int_\Omega u_t v \, d\Omega + \int_\Omega \nabla v \cdot \nabla u \, d\Omega + \underbrace{\int_{\partial\Omega} \ldots \, dS}_{=0} = 0$

because of BCs

choose $u = \sum u_i^{(t)} \varphi_i$, take $v = \varphi_j$

$$\Rightarrow \boxed{A_1 \ddot{\vec{u}} + \beta A_2 \dot{\vec{u}} + A_3 \vec{u} = \vec{0}}$$

$u_t = \sum \dot{u}_i \varphi_i$

$u_{tt} = \sum \ddot{u}_i \varphi_i$

bonus: max 5pt

$$A_1 = \begin{pmatrix} \ddots & & \Theta \\ & \langle \varphi_i, \varphi_j \rangle & \\ \Theta & & \ddots \end{pmatrix}$$

"mass-matrices"

$$A_2 = \begin{pmatrix} \ddots & & \Theta \\ & \ddots & \\ \Theta & & \ddots \end{pmatrix}$$

tridiagonal in 1D with in 2D penta-diagonal

depends on "administration"

gradient

$$A_3 = \begin{pmatrix} \ddots & & \Theta \\ & \langle \nabla\varphi_i, \nabla\varphi_j \rangle & \\ \Theta & & \ddots \end{pmatrix}$$

"stiffness-matrix"

tridiagonal in 1D with in 2D: pentadiagonal

Q6

max 10 pt

max 5pt

a) $\begin{cases} \dot{y} = \sqrt{y} \\ y(0) = 1 \end{cases} \Rightarrow \int \frac{dy}{y} = \int dt$

$2\sqrt{y} = t + c$

$2\sqrt{1} = 0 + c$

$c = 2$

$\Big\} \; 2\sqrt{y} = t + 2$

$\Downarrow$

$y = \left(\frac{1}{2}t + 1\right)^2$

exact sol.

call $y^+ = y^{n+1} = \left(\frac{1}{2}(t + \overset{\Delta t}{h}) + 1\right)^2$

call $y^- = y^{n-1} = \left(\frac{1}{2}(t - \underset{\Delta t}{h}) + 1\right)^2$

$$\Rightarrow \frac{y^+ - y^-}{\Delta t} = \frac{\left(\frac{1}{2}(t+\Delta t)+1\right)^2 - \left(\frac{1}{2}(t-\Delta t)+1\right)^2}{\Delta t}$$

$$= \frac{\left(\frac{1}{4}(t^2 + 2\Delta t\, t + (\Delta t)^2) + 1 + t + \Delta t\right) - \left(\frac{1}{4}(t^2 - 2\Delta t . t + (\Delta t)^2) + 1 + (t - \Delta t)\right)}{\Delta t}$$

$$= \frac{\Delta t . t + 2 . \Delta t}{2 . \Delta t} = \frac{1}{2}t + 1 = \sqrt{y^n}$$

max 5pt

b) nonstandard FD's: 1) denominator function $\phi(\Delta t)$ instead of Δt

$\overset{\shortparallel}{\Delta t} + O((\Delta t)^2)$

2) nonlocal approximation of RHS

example: $u_j^2 \to u_{j+1} u_j$

3) -- ---

Q7

max 15pt

max 3pt (a) $\theta = 0$ EF

$\theta = 1$ EB

$\theta = 1/2$ TM

max 3pt (b) ord = 1 $\theta \neq \frac{1}{2}$, ord = 2 $\theta = \frac{1}{2}$

max 3pt (c) implicit $\theta \neq 0$, explicit $\theta = 0$

max 3pt (d) $f(y) = \lambda y$

$$\Rightarrow y^{n+1} = y^n + \Delta t\left(\theta \lambda y^{n+1} + (1-\theta)\lambda y^n\right)$$

$z = \lambda \Delta t$

$$= y^n + z\theta y^{n+1} + (1-\theta) z y^n$$

$$(1 - z\theta) y^{n+1} = (1 + (1-\theta)z) y^n$$

$$y^{n+1} = \underbrace{\frac{1 + (1-\theta)z}{1 - z\theta}}_{R_\theta(z)} y^n$$

max 3pt (e)

A) RK_2 or Taylor 2, $R_2(z)$

B) EF, $R_0(z)$ ($\theta = 0$)

C) EB, $R_1(z)$ ($\theta = 1$)

D) ExMP (leapfrog). × (no $R(z)$)

E) TM, $R_{1/2}(z)$ ($\theta = \frac{1}{2}$)

F) RK_4 or Taylor 4, $R_3(z)$