Numerical Methods for Time-Dependent PDEs

Spring 2025

Exercises for Lecture 1

Exercise 1.1: Well-posedness

Consider the heat equation in one space dimension:

$$\begin{cases} u_t = \delta u_{xx}, \ 0 < x < 1, t > 0, \delta > 0, \\ u(x,0) = u_0(x), \\ u(0,t) = u(1,t) = 0. \end{cases} \tag{1}$$

(a) show that: $||u(.,t)||_2^2 \leq ||u_0(.)||_2^2$. (i.e., the energy estimate)
(b) using (a), show that the solution of PDE model (1) is unique.
(c) also, show continuity with respect to initial conditions.
Note that the existence of the solution of PDE (1) can be established using Fourier series.

Exercise 1.2: PDE solutions

Consider the potential equation (or Laplace equation) :

$$\Delta u = 0. \tag{2}$$

Identify the points $(x, y) \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$.
(a) Check that $u(x, y) = \Re(f(z))$ is the solution of the potential equation for, for example, $f(z) = 1$, $f(z) = z^2$ and $f(z) = \log(z - z_0)$, $z_0 \in \mathbb{C} \setminus \{0\}$ (hint: polar coordinates).
(b) prove that if $u(x, y)$ and $v(x, y)$ are solutions of the Cauchy-Riemann differential equations:

$$\begin{cases} u_x + v_y = 0, \\ v_x - u_y = 0, \end{cases}$$

then they also solve the potential equation in (2).

Exercise 1.3: PDE classification

Classify each of the PDEs below as either hyperbolic, parabolic, or elliptic, determine the characteristics and transform the equations to canonical form:

(a) $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
(b) $yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$, consider $x \neq y$.
(c) $y^2 u_{xx} + x^2 u_{yy} = 0$, consider $x, y \neq 0$.

Exercise 1.4: Fourier transform method

Use the Fourier transform method to solve the following linear PDEs with initial condition $u(x,0) = u_0(x)$ (Hint: the final answer will include integrals):

(a) $u_t = u_{xx} - 12u$,
(b) $u_t = \kappa u_{xx} + \gamma u_x$.[1]

Exercise 1.5: Fourier series method

Use the Fourier series method (with separation of variables) to solve the linear heat equation

$$u_t = \kappa u_{xx}, \quad \kappa \in \mathbb{R}$$

with $x \in [0,1]$, initial condition $u(x,0) = \sin(\pi x)$ and zero-Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$. Comment on the cases $\kappa < 0$ (backward heat equation) and $\kappa > 0$ (forward heat equation).

Exercise 1.6: Method of characteristics

Solve the PDE:

$$u_t + \frac{1}{1 + \frac{1}{2}\cos(x)} u_x = 0,$$

with initial condition $u(x,0) = u_0(x)$. Show that the solution is given by $u(x,t) = u_0(\xi)$, where ξ is the unique solution of the equation $\xi + \frac{1}{2}\sin(\xi) = x + \frac{1}{2}\sin(x) - t$.

[1] This equation models heat transfer in a long heated bar that is exchanging heat with the surrounding medium.