Numerical Methods for Time-Dependent PDEs

Spring 2025

Exercises for Lecture 6

Exercise 6.1

Show that for the nonlinear hyperbolic PDE

$$\frac{\partial u}{\partial t} + \frac{\partial [F(u)]}{\partial x} = 0 \tag{1}$$

the following property holds:

$$\int_{-\infty}^{\infty} u(x,t)\ dx = \int_{-\infty}^{\infty} u(x,0)\ dx \quad \forall t \geq 0,$$

if we assume that $\lim_{x\to\pm\infty} F(u(x,t)) = 0, \quad \forall t \geq 0$. When we apply a *finite-volume method* to equation (1), we write the approximation in flux-differencing form:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(F_{i+1}^n - F_i^n).$$

Show that the following discrete version of the conservation property holds:

$$\Delta x \sum_{i=I}^{J} u_i^{n+1} = \Delta x \sum_{i=I}^{J} u_i^n - \Delta t(F_{J+1}^n - F_I^n),$$

for all choices of indices I and $J > I$.

Exercise 6.2

Show that for

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty,\ t > 0, \tag{2}$$

a slightly modified version of the upwind method

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n (u_i^n - u_{i-1}^n)$$

is consistent. Argue that this method is also consistent for the two PDEs

$$u_t + (\tfrac{u^2}{2})_x = 0,$$

$$(u^2)_t + (\tfrac{2u^3}{3})_x = 0.$$

Now, we consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{3}$$

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Exercise 6.3

Show, by using d'Alembert's formula, that the solution $u(x,t) = u_0(x + ct)$ of the linear advection PDE

$$u_t - cu_x = 0, \quad u(x,0) = u_0(x), \quad -\infty < x < \infty, \ t > 0$$

is also a solution of the wave equation (3) with the special initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \\ u_t(x,0) &= cu_0'(x). \end{aligned}$$

Apply the first step in the Method-of-Lines to the wave equation (3) for period boundary conditions. Describe the system of ODEs and its stability properties when applying Euler Forward and Euler Backward.

Exercise 6.4

Work out the Von Neumann stability analysis for the wave equation with the CTCS-scheme.

Exercise 6.5

Describe a central second-order FD scheme for the Euler-Bernoulli equation with $b > 0$:

$$u_{tt} = -b^2 u_{xxxx}.$$

This PDE models the vertical motion of a thin horizontal beam with small displacements from the rest position. Show that for stability we must have

$$b \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}.$$