**Numerical Methods for Time-Dependent PDEs**

**Spring 2025**

Exercises for Lecture 7

# Exercise 7.1

Consider the logistic ODE model:

$$\dot{u} = u - u^2$$

with initial condition $u(0) = u^0$. First, check that the exact solution satisfies:

$$u(t) = \frac{u^0}{u^0 + (1 - u^0)\mathrm{e}^{-t}}.$$

Show that we obtain, from this expression, the following *exact* finite-difference scheme:

$$\frac{u^{n+1} - u^n}{[1 - \mathrm{e}^{-\Delta t}]} = u^{n+1}(1 - u^n).$$

# Exercise 7.2

Verify that the scheme:

$$\left\{ \begin{array}{l} \frac{u^{n+1} - u^n}{\mathrm{e}^{\pi \Delta t} - 1} = u^n, \quad n = 0, 1, 2, ...; \Delta t > 0, \\ \\ u^0 = 1, \\ \\ u^n \approx u(t^n) = u(n\Delta t), \end{array} \right\}$$

is an *exact* finite difference (FD) scheme for the ODE:

$$\left\{ \begin{array}{l} \dot{u}(t) = \pi \; u(t), \\ u(0) = 1. \end{array} \right\}$$

# Exercise 7.3

(a) Check that the *Leapfrog* method

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = \sqrt{u^n}, \quad u^0 = 1, \; u^1 = \frac{1}{4}(\Delta t)^2 + \Delta t + 1$$

is an *exact* finite difference (FD) scheme for: $\quad \dot{u}(t) = \sqrt{u(t)}$ with $u(0) = 1$.

(b) Give two important ingredients of a *nonstandard* FD scheme, when compared to a standard FD scheme.

## Exercise 7.4

Consider the nonlinear ODE model:

$$\dot{u} = u^2 - u^3$$

with initial condition $u(t^0) = u^0$. Derive the *nonstandard* finite-difference scheme:

$$u^{n+1} = \frac{(1 + 2\phi(\Delta t)u^n)u^n}{1 + \phi(\Delta t)(u^n + (u^n)^2)},$$

by making the non local approximations: $u^2 \to 2\,(u^n)^2 - u^{n+1}u^n$ and $u^3 \to u^{n+1}\,(u^n)^2$ Which function $\phi(\Delta t)$ would be a good choice?

## Exercise 7.5

Consider Fisher's PDE

$$u_t = u_{xx} + u(1-u).$$

Derive the *non-standard* finite-difference scheme with the *nonlocal* approximation[1]

$$2\bar{u}_i^n - u_i^{n+1} - \bar{u}_i^n u_i^{n+1}$$

for the reaction term. Use the standard FT and CS approximations for $u_t$ and $u_{xx}$, respectively. You may assume $\frac{\Delta t}{(\Delta x)^2} = \frac{1}{2}$.

The solution $u(x,t)$ satisfies "the boundedness condition":

$$0 \leq u(x,0) \leq 1 \Rightarrow 0 \leq u(x,t) \leq 1, \;\; \forall t > 0.$$

Prove a similar statement for the difference scheme:

$$0 \leq u_i^0 \leq 1 \Rightarrow 0 \leq u_i^n \leq 1, \quad \forall n \geq 1, \; \forall \text{ relevant } i.$$

[1] $\bar{u}_i^n := \frac{u_{i+1}^n + u_i^n + u_{i-1}^n}{3}$.