Lecture 1

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Numerical Methods for PDEs

Outline of Lecture 1

- organization of the course
- wellposedness: Hadamard, 1902
- classification & method of characteristics, *canonical form
- time-dependent PDEs and application areas
- Fourier series method
- Fourier transform method
- outlook to lecture 2

Organization of the course

- webpage
- "location" and "hours"
- communication
- *final exam* (70%) $\leftrightarrow$ exercises in lectures (check the webpage!)
- *assignments* (5% C1A, 10% C1B, 15% C2)
- programming: Matlab, Python

Well-posed vs ill-posed [1]

The problem of reconstructing the image of an object and its surroundings is ill-posed (i.e., there is no uniqueness or stability of solutions), our brain is capable of solving it rather quickly.
This is due to the brain's ability to use its extensive previous experience (a priori information).
A quick glance at a person is enough to determine if he or she is a child or a senior, but it is usually not enough to determine the person's age with an error of at most five years.

Well-posed vs ill-posed [2]

What are inverse and ill-posed problems?
While there is no universal formal definition for inverse problems, an "ill-posed problem" is a problem that either has no solutions in the desired class, or has many (two or more) solutions, or the solution procedure is unstable (i.e., arbitrarily small errors in the measurement data may lead to indefinitely large errors in the solutions).
Most difficulties in solving ill-posed problems are caused by the solution instability. Therefore, the term "ill-posed problems" is often used for unstable problems.

Well-posed vs ill-posed [3]

What is an ill-posed problem?

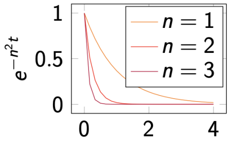
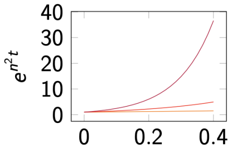
A problem is ill-posed if it does not satisfy the 3 conditions of a well-posed problem:

- ***Existence***: There exists a solution.
- ***Uniqueness***: The solution is unique.
- ***Stability***: The solution depends continuously on initial conditions.

The inverse of a well-posed problem is generally ill-posed.

Well-posed vs ill-posed [4]

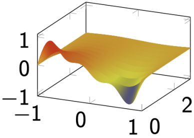
Parabolic PDEs: well-posed vs. ill-posed

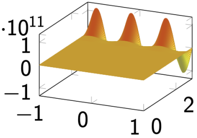
Heat equation	(Reversed time)
$u_t - u_{xx} = 0$	$u_t + u_{xx} = 0$
$u(x,0) = f(x)$	$u(x,0) = f(x)$
$u(0,t) = u(\pi,t) = 0$	$u(0,t) = u(\pi,t) = 0$
$\sum_{n=1}^{\infty} f_n sin(nx) e^{-n^2 t}$	$\sum_{n=1}^{\infty} f_n sin(nx) e^{n^2 t}$
	
Existence? ✓ Uniqueness? ✓ Stability? ✓	Existence? × Uniqueness? ✓ Stability? ×

Well-posed vs ill-posed [5]

Parabolic PDEs: well-posed vs. ill-posed - cont.

Given the Fourier series we found for $f(x) = -x, [-1, 1]$ at $k = 3$, here's what the graph of $u(x, t)$ looks like:

<table>
<tr><td>Heat equation</td><td>(Reversed time)</td></tr>
<tr><td>$\boldsymbol{u_t - u_{xx} = 0}$


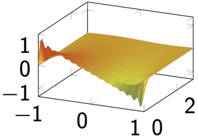
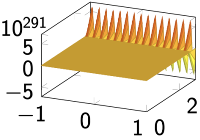
A reasonable approximation</td><td>$\boldsymbol{u_t + u_{xx} = 0}$


Huh...?</td></tr>
</table>

Well-posed vs ill-posed [6]

Why heat equation with reverse time is ill-posed

As we increase k, one graph becomes more accurate, while the other becomes more and more chaotic. $k = 15$:

Heat equation	(Reversed time)
$\boldsymbol{u_t - u_{xx} = 0}$	$\boldsymbol{u_t + u_{xx} = 0}$
	
More accurate	Chaos!

Well-posed vs ill-posed [7]

Well-posed problems	Ill-posed problems
Arithmetic	
Multiplication by a small number A $Aq = f$	Division by a small number $q = A^{-1}f \quad (A \ll 1)$
Algebra	
Multiplication by a matrix $Aq = f$	$q = A^{-1}f$, A is an ill-conditioned, degenerate or rectangular $m \times n$-matrix
Calculus	
Integration $f(x) = f(0) + \int_0^x q(\xi)\, d\xi$	Differentiation $q(x) = f'(x)$
Differential equations	
The Sturm–Liouville problem $u''(x) - q(x)u(x) = \lambda u(x)$, $u(0) - hu'(0) = 0$, $u(1) - Hu'(1) = 0$	The inverse Sturm–Liouville problem. Find $q(x)$ using spectral data $\{\lambda_n, \lVert u_n \rVert\}$

FDs versus FEs

In this course:

* Finite differences & finite volumes

  ↓ conservation laws

  based on:
  - Taylor expansions
  - eigenvalues
  - time-integration methods

* parabolic and hyperbolic PDEs (time-dependent)

[* finite elements ⟶ based on variational formulations
(functional analysis plays an important role)

- elliptic PDEs
(time-independent; stationary)]

Notation and definitions

Partial Differential Equations : PDEs

notation: $u_x = \dfrac{\partial u}{\partial x} = \partial_x u$, $u_{xx} = \dfrac{\partial^2 u}{\partial x^2} = \partial_{xx} u = \partial_x^2 u = \dots$

$$u_{4x} = u_{xxxx} = \frac{\partial^4 u}{\partial x^4} = \dots$$

general form of PDE: $F(x, y, z, t, u, u_x, u_y, u_t, u_{xx}, \dots) = 0$

a given "operator" (function)

Independent variables: $x, y, z, t, \dots$

Dependent variables: $u, v, P, T, \rho, \dots$

Linear PDE: $L\,u(\vec{x}) = f(\vec{x})$ with $L(a.u + b.v) = a.L(u) + b.L(v)$

$f = 0 \Rightarrow$ homogeneous PDE, else: inhomogeneous

if not linear, then: quasi-linear ---- non-linear

Example: $u_{xy} = 0 \xrightarrow{\text{integrating twice}} u(x,y) = g(y) + f(x)$ arbitrary functions!

Boundary conditions

a) Dirichlet BC (on $\partial\Omega$): $u\Big|_{\partial\Omega} = \text{given}$

boundary of spatial domain

b) Neumann BC : $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = \text{given}$

c) Robin BC : $\frac{\partial u}{\partial n} + a.u \Big|_{\partial\Omega} = \text{given}$

d) periodic BC (example) : $u(0,t) = u(1,t)$, if $\Omega = [0,1]$

e) absorbing BC

f) flux BC

g) transport BC

etcetera ....

In general: "count the number of derivatives in PDE" $\leftrightarrow$ "how many conditions needed"
($\approx$ "highest" derivative")

Time-dependent PDEs : $\begin{cases} \text{boundary conditions (BCs)} \\ \text{initial condition(s) (IC(s))} \end{cases}$

Basic (linear) examples [1]

1) Laplace equation (stationary): $\Delta u = 0$
2) Poisson equation (" "): $\Delta u = f$
3) Helmholtz equation (" "): $\Delta u + k^2 u = 0$
4) wave equation: $u_{tt} - c^2 \Delta u = 0$
5) heat or diffusion equation: $u_t - \kappa . \Delta u = 0$
6) advection equation: $u_t + a u_x = 0$
7) telegraph equation (transmission lines): $u_{tt} + a u_t + bu = u_{xx}$
8) Klein-Gordon equation (relativistic electrons): $\Box u + \gamma u = 0$

$\Box = \frac{\partial^2}{\partial t^2} - c^2 \Delta$ ("d'Alembert operator")

9) Boussinesq equation (hydrodynamics): $u_{tt} - \alpha^2 \Delta u - \beta^2 \Delta u_{tt} = 0$

mixed derivatives!

10) Bi-harmonic wave equation (elasticity): $u_{tt} + c^2 \Delta^2 u = 0$

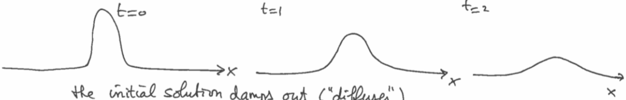
fourth-derivatives!

Basic (linear) examples [2]

$\boxed{\text{Heat equation}}$ ("diffusion equation")

linear, first/second order, parabolic

$$\boxed{\begin{cases} u_t = K \cdot u_{xx} & \\ u(x,0) = u_0(x) & \text{IC} \end{cases}}$$



the initial solution damps out ("diffuses")

example of a solution: $u(x,t) = \dfrac{e^{-\frac{x^2}{1+4Kt}}}{\sqrt{1+4Kt}}$

"backward" heat equation : $u_t = -u_{xx}$ <u>ill</u>-posed

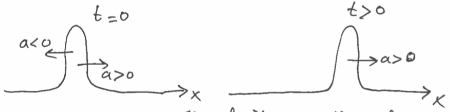
Basic (linear) examples [3]

Advection equation ("transport equation", "one-way wave equation")

linear, first-order, "hyperbolic"

$$\boxed{\begin{cases} u_t + a\,u_x = 0 \\ u(x,0) = u_0(x) \qquad \text{IC} \end{cases}}$$

solution: $u(x,t) = u_0(x-at)$



wave moves with velocity a, without changing its shape

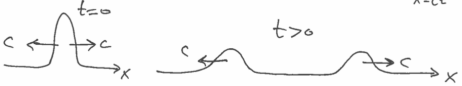
Basic (linear) examples [4]

Wave equation ("two-way" wave equation)

linear, second-order, hyperbolic

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases} \Bigg\} \text{ICs}$$

$\Rightarrow$ d'Alembert solution: $u(x,t) = \frac{1}{2}\left[g(x-ct) + g(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi$



"backward" wave equation: $u_{tt} + u_{xx} = 0$ ill-posed as an initial-value problem

however: $u_{xx} + u_{yy} = 0$ + BCs (elliptic) is well-posed

Some nonlinear examples [1]

Burgers' equation

non-linear, second/first-order, parabolic/hyperbolic

$$\boxed{u_t = \varepsilon u_{xx} - u u_x}$$

$\varepsilon = 0$: inviscid Burgers eq.

$\varepsilon > 0$: viscous Burgers eq.

$\varepsilon = 0$ (hyperbolic) : shock waves are possible (depending on IC)

(classical solution does not exist anymore ...)

$\varepsilon > 0$ (parabolic) and $t > 0$:

steeper and steeper (but will never "break")

Some nonlinear examples [2]

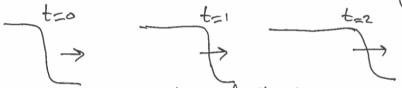
$\boxed{\text{Fisher equation}}$

nonlinear, parabolic

$$\boxed{u_t = u_{xx} + u(1-u)}$$

Travelling wave solutions are possible:

(example) $u(x,t) = \dfrac{1}{\left(1 + e^{\frac{x-ct}{\sqrt{6}}}\right)^2}, \quad c = \dfrac{5}{\sqrt{6}}$



constant velocity c

waves do not change shape (see advection equation)
↑linear

note that this is a nonlinear PDE!

Some nonlinear examples [3]

$\boxed{\text{Korteweg-de Vries equation}}$ "KdV"

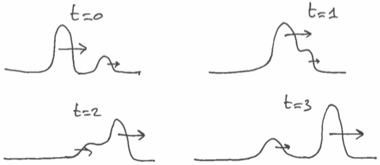
$$\boxed{\underset{\text{nonlinear}}{u_t + 6\,u u_x} + \underset{\text{third order}}{u_{xxx}} = 0}$$

type = ?

<u>soliton</u> solutions : $u(x,t) = \frac{c}{2}\,\text{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - x_0)\right)$

$\left(\text{sech} = \frac{1}{\cosh}\right)$

<u>Non</u>linear superposition of waves!



Classification [1]

Two independent variables

$$\boxed{\begin{array}{l} 2^{nd} \text{ order linear PDE}: \; A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u + G = 0 \\ \quad \text{suppose first}: \; A, B, C, \cdots \text{ are constants} \end{array}}$$

Remember: 2^{nd} order algebraic equation: $\underbrace{ax^2+bxy+cy^2}_{\text{"principal part"}} + dx + ey + f = 0$

these are curves in the x-y plane

if $b^2-4ac>0$, then the curve is a hyperbola

if $b^2-4ac=0$, then the curve is a parabola

if $b^2-4ac<0$, then the curve is an ellipse

Applying a suitable transformation of variables $\Rightarrow$ "normal form"

$$\begin{cases} \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \\ x^2 = y \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \end{cases}$$

Classification [2]

"principal part" of PDE: $A u_{xx} + B u_{xy} + C u_{yy}$

if $B^2 - 4AC > 0$, then the PDE is hyperbolic
if $B^2 - 4AC = 0$, then the PDE is parabolic
if $B^2 - 4AC < 0$, then the PDE is elliptic

linear PDE with variable coefficients: $A, B, C, \dots$ depend on x and y

Check $B^2(x,y) - 4A(x,y)C(x,y)$

if $B^2(x,y) - 4A(x,y)C(x,y) > 0$ at (x,y), then the PDE is hyperbolic at (x,y)

if " " " $= 0$ at (x,y), then the PDE is parabolic at (x,y)

if " " " < 0 at (x,y), then the PDE is elliptic at (x,y)

Note: first-order derivatives and the inhomogeneous term in this PDE do not play a role in this classification!

Classification [3]

Examples:

① $u_{xx} + u_{yy} = 0$

$A = 1,\ B = 0,\ C = 1 \Rightarrow B^2 - 4AC = -4 < 0$

$\Rightarrow$ PDE is elliptic

② $u_y = u_{xx}$

$A = -1,\ B = 0,\ C = 0 \Rightarrow B^2 - 4AC = 0$

$\Rightarrow$ PDE is parabolic

③ $u_{xx} + x\, u_{yy} = 0 \quad (x \neq 0)$

$A = 1,\ B = 0,\ C = x \Rightarrow B^2 - 4AC = -4x$

$\Rightarrow$ PDE is $\begin{cases} \text{hyperbolic for } x < 0 \\ \text{elliptic for } x > 0 \end{cases}$

Three or more independant variables:

example ④ $\Delta u = u_{xx} + u_{yy} + u_{zz} = 0 \Rightarrow \underbrace{\lambda_1 = \lambda_2 = \lambda_3 = 1 > 0}_{\text{all eigenvalues have the same sign}} \Rightarrow$ elliptic PDE

⑤ $u_z = u_{xx} + u_{yy} \Rightarrow \lambda_1 = \lambda_2 = 1 > 0,\ \lambda_3 = 0 \Rightarrow$ parabolic PDE

⑥ $u_{zz} = u_{xx} + u_{yy} \Rightarrow \lambda_1 = \lambda_2 = 1 > 0,\ \lambda_3 < 0 \Rightarrow$ hyperbolic PDE

Method of characteristics [1]

linear first-order PDEs

consider: $\boxed{a(x,y)z_x + b(x,y)z_y + c(x,y)z = d(x,y)}$

$\underbrace{a,\ b,\ c,\ d}_{\text{given functions}}$

re-write: $a(x,y)z_x + b(x,y)z_y = -c(x,y)z + d(x,y)$

observe: $a(x,y)z_x + b(x,y)z_y = \nabla z \cdot \begin{pmatrix} a \\ b \end{pmatrix} = D_{\binom{a}{b}} z$ (directional derivative)

note: $\begin{pmatrix} a \\ b \end{pmatrix}$ is a vector field in the x-y plane.

$$\boxed{\text{ODE system: } \begin{cases} \dfrac{dx}{dt} = a(x,y) \\[2ex] \dfrac{dy}{dt} = b(x,y) \end{cases}} \Rightarrow \begin{cases} x(t) = \ldots \\ y(t) = \ldots \end{cases} \quad \text{family of curves}$$

tangent vector at these curves: $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}$ (is in the direction of vector $\begin{pmatrix} a \\ b \end{pmatrix}$)

Method of characteristics [2]

Take the derivative of z along these curves:

$$\frac{dz}{dt} = \frac{dz}{dt}(x(t),y(t)) = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = z_x \cdot a(x,y) + z_y \cdot b(x,y)$$

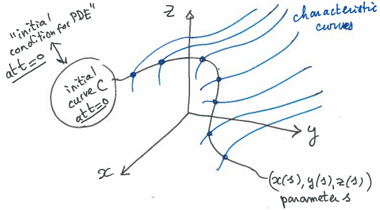
chain rule

$$\xrightarrow{\text{PDE}} = -c(x,y)\,z + d(x,y).$$

$$= -c(t)\,z(t) + d(t)$$

solve... $\Longrightarrow$ $z(t) = e^{-\int_0^t c(\tilde{t})d\tilde{t}} \cdot \left[\int_0^t e^{\int_0^\tau c(\tilde{t})d\tilde{t}} \cdot d(\tau)\,d\tau + z(0)\right]$ (using an "integrating factor")

$\leadsto$ "Method of characteristics" (... "characteristic curves" $x(t)$, $y(t)$)



family of characteristic curves connected to the points of curve C:

$$\begin{cases} x(s,t) = \cdots \\ y(s,t) = \cdots \\ z(s,t) = \cdots \end{cases}$$

at $t=0$: $\begin{cases} x(s,0) = x(s) \\ y(s,0) = y(s) \\ z(s,0) = z(s) \end{cases}$

Method of characteristics [3]

if Jacobian $J = x_s y_t - x_t y_s \neq 0$, then we can invert $x = x(s,t)$, $y = y(s,t)$

to give s and t as functions of x and y : $s = s(x,y)$, $t = t(x,y)$

$\Rightarrow z = z(x,y) = z(s(x,y), t(x,y))$ solves the original PDE

and satisfies the initial condition !

(unique solution)

as before

$$\boxed{\begin{aligned} z(s,t) &= \frac{1}{\mu(s,t)}\left[\int_0^t \mu(s,t)d(s,t)dt + g(s)\right] \\ \mu(s,t) &= \exp\left[\int_0^t c(s,t)dt\right]\end{aligned}}$$

Example :

$$\begin{cases} \dfrac{\partial z}{\partial y} + c \cdot \dfrac{\partial z}{\partial x} = 0 \\ z(x,0) = u_0(x) \end{cases}$$

constant

given

1) find characteristic curves : parameterize initial curve C $\begin{cases} x = s \\ y = 0 \\ z = u_0(s) \end{cases}$

family of characteristic curves satisfy : $\begin{cases} \dfrac{dx}{dt}(s,t) = c \\ \dfrac{dy}{dt}(s,t) = 1 \end{cases}$

solve : (integrate) $\begin{cases} x(s,t) = ct + c_1(s) \\ y(s,t) = t + c_2(s) \end{cases}$

Method of characteristics [4]

2) apply initial conditions: $\begin{cases} x(s,0) = s \\ y(s,0) = 0 \end{cases} \Rightarrow \begin{cases} c_1(s) = s \\ c_2(s) = 0 \end{cases}$

$$\Rightarrow \begin{cases} x(s,t) = ct + s \\ y(s,t) = t \end{cases}$$

---

3) here, we have $c(x,y) = 0$ and $d(x,y) = 0$

$\Rightarrow d(s,t) = 0,\ \mu(s,t) = 1$ (see exp(...))

note: $z(\underset{t=0}{\underset{\uparrow}{x(s,0)}}, \underset{t=0}{\underset{\uparrow}{y(s,0)}}) = z(s,0) = u_0(s) \Rightarrow \begin{cases} x(s,t) = ct + s \\ y(s,t) = t \\ z(s,t) = u_0(s) \end{cases}$

---

4) $z(s,t)$ "$\rightarrow$" $z(x,y)$: $\begin{cases} s = x - ct = x - cy \\ t = y \end{cases}$

$$\Rightarrow z(x,y) = z(s(x,y), t(x,y)) = u_0(x - cy)$$

*Canonical form [1]

$$\boxed{A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u + G = 0}$$

transform $(x,y) \longmapsto (\xi,\eta)$: $\begin{cases} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{cases}$

Jacobian $J = \det \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \neq 0 \leftarrow$ assume

chainrule

$\Longrightarrow$ $u_x = u_\xi \xi_x + u_\eta \eta_x$

$u_y = u_\xi \xi_y + u_\eta \eta_y$

$u_{xx} = u_{\xi\xi} \xi_x^2 + 2 u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$

$u_{yy} = u_{\xi\xi} \xi_y^2 + 2 u_{\xi\eta} \xi_y \eta_y + u_{\eta\eta} \eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$

$u_{xy} = u_{\xi\xi} \xi_y \xi_x + u_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) + u_{\eta\eta} \eta_x \eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$

substitute

$$\Rightarrow \begin{cases} \tilde{A}(\xi_x,\xi_y) u_{\xi\xi} + \tilde{B}(\xi_x,\xi_y,\eta_x,\eta_y) u_{\xi\eta} + \tilde{C}(\eta_x,\eta_y) u_{\eta\eta} = F(\xi,\eta,u,u_\xi,u_\eta) \\ \tilde{A} = A \xi_x^2 + B \xi_x \xi_y + C \xi_y^2 \\ \tilde{B} = 2A \xi_x \eta_x + B (\xi_x \eta_y + \xi_y \eta_x) + 2C \xi_y \eta_y \\ \tilde{C} = A \eta_x^2 + B \eta_x \eta_y + C \eta_y^2 \end{cases}$$

$\Longrightarrow \tilde{B}^2 - 4\tilde{A}\tilde{C} = \underbrace{(\xi_x \eta_y - \xi_y \eta_x)^2}_{>0} (B^2 - 4AC)$

$\Rightarrow$ same classification as before transformation !

*Canonical form [2]

Determine "special" ξ and η such that we obtain the simplest possible form

case 1 : $B^2 - 4AC > 0$ (hyperbolic) $\Rightarrow$ equation $A\alpha^2 + B\alpha + C = 0$ has two real and distinct roots

say: $\alpha = \lambda_1$ and $\alpha = \lambda_2$

choose ξ and η such that $\xi_x = \lambda_1 \xi_y$ and $\eta_x = \lambda_2 \eta_y$ (*)

$$\Rightarrow \begin{cases} \tilde{A} = A.\lambda_1^2 \xi_y^2 + B\lambda_1^2 \xi_y^2 + C\xi_y^2 \\ \quad = \xi_y^2 \underbrace{(A\lambda_1^2 + B\lambda_1 + C)}_{=0} = 0 \\ \text{similarly: } \tilde{C} = 0 \end{cases}$$

$$\tilde{B}^2 = (B^2 - 4AC)(\xi_x \eta_y - \xi_y \eta_x)^2 > 0$$

Note: (*)'s are first-order linear PDEs in ξ and η

$$\xrightarrow{\text{MofCh}} \begin{cases} x_t = 1 \\ y_t = -\lambda_1 \end{cases} \text{ and } \begin{cases} x_t = 1 \\ y_t = -\lambda_2 \end{cases} \Rightarrow \begin{cases} \dfrac{dy}{dx} + \lambda_1 = 0 \text{ and } \dfrac{dy}{dx} + \lambda_2 = 0 \end{cases}$$

characteristic curves are : $f_1(x,y) = c_1$ and $f_2(x,y) = c_2$

divide transformed PDE (previous page) by $\tilde{B} > 0$ $\Rightarrow$ canonical form (case 1) $\boxed{u_{\xi\eta} = \varphi(\xi, \eta, u, u_\xi, u_\eta)}$

*Canonical form [3]

Case 2 : $B^2 - 4AC = 0 \Rightarrow \tilde{A} = \tilde{B} = 0, \tilde{C} \neq 0$

canonical form (case 2)

$\Rightarrow \boxed{u_{\eta\eta} = \varphi(\xi, \eta, u, u_\xi, u_\eta)}$

Case 3 : $B^2 - 4AC < 0 \overset{\cdots}{\Rightarrow} \boxed{u_{\alpha\alpha} + u_{\beta\beta} = \psi(\alpha, \beta, u, u_\alpha, u_\beta)}$

complex roots ---

with $\begin{cases} \alpha = \frac{1}{2}(\xi + \eta) \\ \beta = \frac{1}{2i}(\xi - \eta) \end{cases}$

canonical form (case 3)

<u>Example</u> : $u_{xx} = x^2 u_{yy}$

$A = 1,\ B = 0,\ C = -x^2 \Rightarrow \begin{cases} y + \frac{1}{2}x^2 = c_1 \\ y - \frac{1}{2}x^2 = c_2 \end{cases}$, choose $\begin{cases} \xi = y + \frac{1}{2}x^2 \\ \eta = y - \frac{1}{2}x^2 \end{cases}$

$$\overset{\cdots\cdots}{\Longrightarrow} 4x^2 u_{\xi\eta} = u_\xi - u_\eta$$

$$\Leftrightarrow 4(\xi - \eta) u_{\xi\eta} = u_\xi - u_\eta$$

$$\Leftrightarrow \boxed{u_{\xi\eta} = \frac{1}{4(\xi - \eta)}(u_\xi - u_\eta)}$$

PDE examples [1]: Fourier and linear PDEs [1]

The most basic of all problems involving partial differential equations are linear PDEs with constant coefficients posed on unbounded domains. Such problems are translation-invariant, and as a result, their solutions can be found by the Fourier transform.

For example, here are three linear constant-coefficient equations in one space variable:

$$u_t = u_x, \qquad u_t = -u_{xx} - u_{xxxx}, \qquad u_t = u_{xxxx}. \tag{1}$$

Inserting the ansatz $u(x,t) = \exp(\mathrm{i}kx + f(k)t)$ gives a relation between k and $f(k)$—the *dispersion relation*,

$$f(k) = \mathrm{i}k, \qquad f(k) = k^2 - k^4, \qquad f(k) = k^4.$$

The corresponding solutions for real k are

$$u(x,t) = e^{\mathrm{i}kx+\mathrm{i}kt}, \qquad u(x,t) = e^{\mathrm{i}kx+(k^2-k^4)t}, \qquad u(x,t) = e^{\mathrm{i}kx+k^4t}. \tag{2}$$

Fourier analysis tells us that in the space L^2 defined by the norm $\|u\| = (\int_{-\infty}^{\infty} |u(x)|^2\,\mathrm{d}x)^{1/2}$, all solutions to (1) can be obtained as superpositions of the solutions (2):

$$u(x,t) \;=\; \int_{-\infty}^{\infty} \hat{u}(k,t)e^{\mathrm{i}kx}\,\mathrm{d}k \;=\; \int_{-\infty}^{\infty} \hat{u}(k,0)e^{\mathrm{i}kx+f(k)t}\,\mathrm{d}k, \tag{3}$$

where $\hat{u}(k,t)$ denotes the Fourier transform of $u(x,t)$ with respect to x. In other words, $\hat{u}(k,t)$ evolves for each k according to the trivial ordinary differential equation $\hat{u}_t = f(k)\hat{u}$ with solution $\hat{u}(k,t) = \exp(f(k)t)\hat{u}(k,0)$. Thus we see that for linear equations with constant coefficients on unbounded domains, when we take the Fourier transform,

- *Differential operators become polynomials in* k, and
- *The PDE becomes an uncoupled system of ODEs, one ODE for each* k.

[1] ® Nick Trefethen, Oxford University

PDE examples [2]: forward heat equation

$$\boxed{u_t = \Delta u.}$$

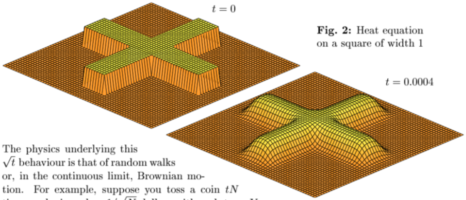


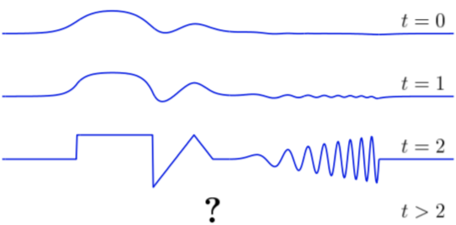
Fig. 2: Heat equation on a square of width 1

The physics underlying this $\sqrt{t}$ behaviour is that of random walks or, in the continuous limit, Brownian motion. For example, suppose you toss a coin tN times and win or lose $1/\sqrt{N}$ dollars with each toss. Your profit follows a binomial distribution that converges to the Gaussian $e^{-x^2/4t}/\sqrt{4\pi t}$ in the limit $N \to \infty$. Arbitrarily large profits are possible, but anything much bigger than $\sqrt{t}$ is very unlikely. This $\sqrt{t}$ effect is at the root of much of the field of statistics, and it was the basis of Einstein's epochal paper on Brownian motion in his *annus mirabilis* 1905.

PDE examples [3]: backward heat equation

$$\boxed{u_t = -\Delta u.}$$

Fig. 1: Loss of smoothness



PDE examples [4]: wave equation

$$\boxed{u_{tt} = \Delta u,}$$

The wave equation describes linear, nondispersive wave propagation. For example, Figure 1 presents a pair of images that show the outward spread of a circular pulse in 2D. At $t = 0$ we begin with a cone of radius 0.1 with $u_t(0) = 0$. At $t = 2$, the cone has spread to a concentric ring of outer radius exactly 2.1.

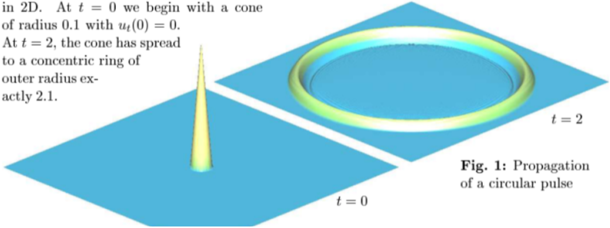


Fig. 1: Propagation of a circular pulse

PDE examples [5]: wave equation in 1d

In one dimension the wave equation (→ *ref*) takes the form

$$\boxed{u_{tt} = u_{xx},} \tag{1}$$

the simplest second order hyperbolic PDE. The standard example of a physical system governed by the wave equation is a vibrating ideal elastic string (such as a guitar string) fixed at both ends. If the string is distorted, or plucked, at some initial time and then allowed to vibrate, the displacement of the resulting transverse wave will be a solution of (1). This equation also models many other physical problems, such as propagation of sound waves in a tube.

An initial value problem can be posed by combining (1) with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

The unique solution to this problem can be expressed by d'Alembert's formula,

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,\mathrm{d}y.$$

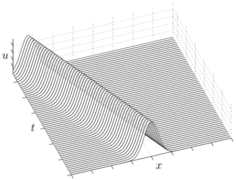


Fig. 1: Propagation in a single direction

Alternatively, for any initial data, solutions to (1) can be written as a linear combination

$$u(x,t) = F(x+t) + G(x-t),$$

where F represents a left-going and G a right-going wave. D'Alembert's solution is the special case in which the left-going and right-going waves are

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2}\int_0^x g(y)\,\mathrm{d}y,$$

$$G(x) = \frac{1}{2}f(x) - \frac{1}{2}\int_0^x g(y)\,\mathrm{d}y.$$

PDE examples [6]: beam equation

$$\boxed{u_{tt} = -u_{xxxx}.}$$

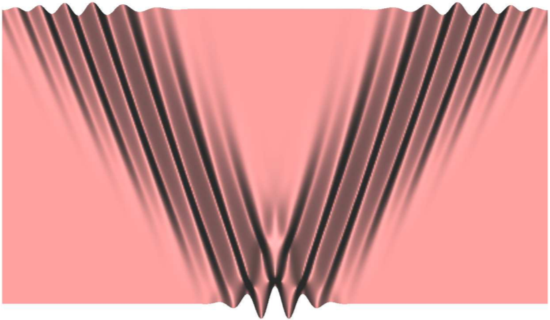


Fig. 1: Wave propagation with $c = \pm 4$, $c_g = \pm 8$

PDE examples [7]: Schrödinger equation

The Schrödinger equation, the basis of quantum mechanics, was discovered by Erwin Schrödinger during his skiing holiday at the end of 1925 and analyzed by him in a series of papers published in *Annalen der Physik* in 1926. By the end of that year, the face of physics had changed. Schrödinger won the Nobel Prize in Physics in 1933.

$$i\hbar \frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[-\frac{\hbar}{2m}\Delta + V(\mathbf{r})\right]\Psi(\mathbf{r},t) \tag{1}$$

where $i = \sqrt{-1}$, Ψ is called the *wave function* and $\hbar$ is Planck's constant divided by 2π. Since our convention in this book is to take u as the dependent variable and strip away constants, we shall take the *time-dependent Schrödinger equation* instead to be

$$\boxed{iu_t = [-\Delta + V(\mathbf{r})]u.} \tag{2}$$

PDE examples [8]: Gray-Scott model

The *Gray–Scott equations* were formulated originally by Gray and Scott in 1983; we shall not discuss their original chemical motivation:

$$\boxed{u_t = \epsilon_1 \Delta u - uv^2 + F(1-u), \qquad v_t = \epsilon_2 \Delta v + uv^2 - (k+F)v.} \tag{1}$$

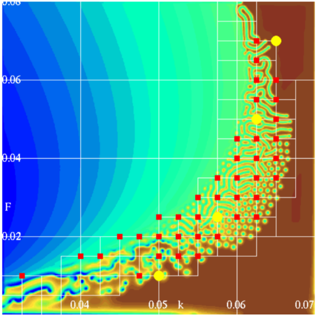


Fig. 2. Dependence on k and F.

PDE examples [9]: Korteweg-de Vries equation

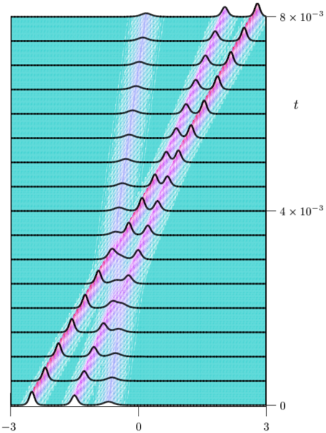


Fig. 1: Soliton interactions

$$\boxed{u_t + uu_x + u_{xxx} = 0.}$$

PDE examples [10]: sine-Gordon model

$$\boxed{u_{tt} - u_{xx} = \sin u.}$$

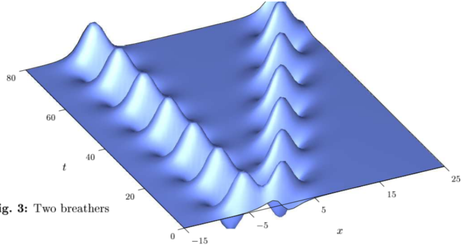


Fig. 3: Two breathers

PDE examples [11]: compacton equations

$$\boxed{u_t + (u^m)_x + (u^n)_{xxx} = 0.}$$

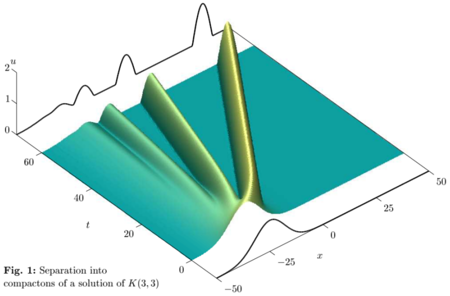


Fig. 1: Separation into compactons of a solution of $K(3,3)$

PDE examples [12]: Boussinesq equation

$$\boxed{u_{tt} - u_{xx} = u_{xxxx} + (u^2)_{xx},}$$

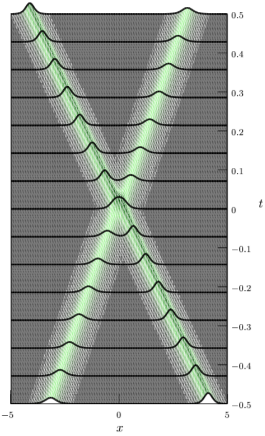


Fig. 2: soliton interactions

PDE examples [13]: blow-up with u^p nonlinearity

$$\boxed{u_t = u_{xx} + u^p}$$

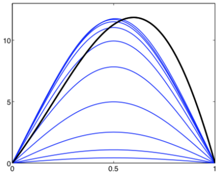


Fig. 2: $\alpha = 6.84$: decay to 0 ($t = 0, 0.1, 0.2, \ldots, 1$ from top to bottom)

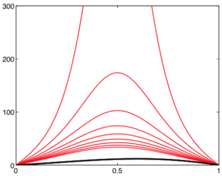


Fig. 3: $\alpha = 6.85$: blow-up ($t = 0, 0.8, 0.805, 0.810, \ldots, 0.840$ from bottom to top)

PDE examples [14]: blow-up with e^u nonlinearity

$$\boxed{u_t = u_{xx} + \lambda e^u}$$

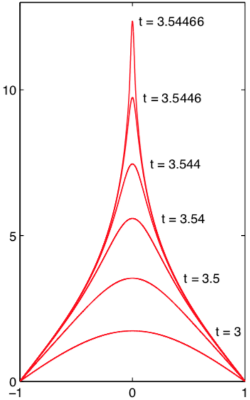


Fig. 1: blowup at $t \approx 3.54466$ with $\lambda = 1$ and zero initial data.

PDE examples [15]: advection-diffusion

$$\boxed{u_t + \mathbf{a} \cdot \nabla u = \varepsilon \Delta u.}$$

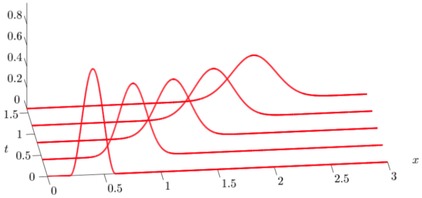


Fig. 1: Solution with $a(x,t) = 1$, $\varepsilon = 10^{-2}$

PDE examples [16]: porous media

$$\boxed{u_t = \Delta(u^m).}$$

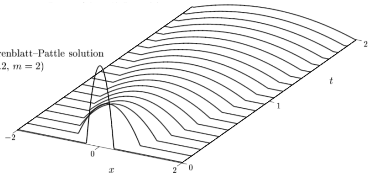


Fig. 1: 1D Barenblatt–Pattle solution ($\Gamma = 0.2$, $m = 2$)

PDE examples [17]: Fisher equation

$$u_t = Du_{xx} + ru\left(1 - \frac{u}{K}\right).$$

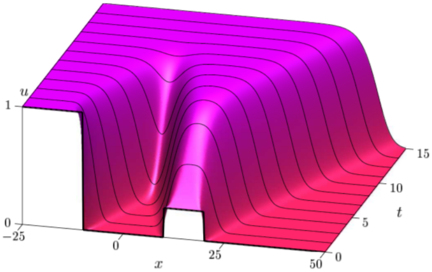


Fig. 1: Formation of traveling wave

PDE examples [18]: Allen-Cahn equation

$$\boxed{u_t = u_{xx} + u - u^3.}$$

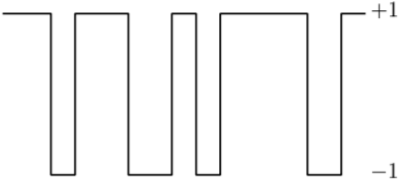


Fig. 1: Metastable fronts (schematic)

PDE examples [19]: Cahn-Hilliard equation

$$\boxed{u_t = \Delta(u^3 - u) - \varepsilon \Delta^2 u.}$$

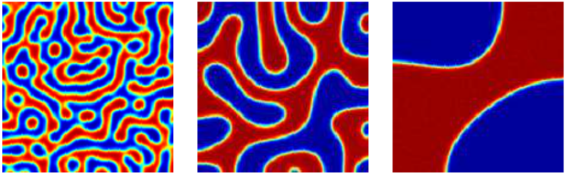


Fig. 1: Solutions in 2D for small, medium, and large t

PDE examples [20]: Perona-Malik model

$$\boxed{u_t = \nabla \cdot \left(g(|\nabla u|)\, \nabla u \right),}$$

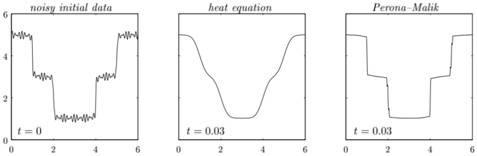
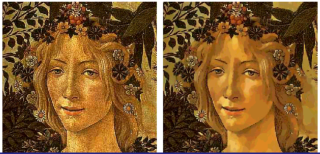


Fig. 1: Edge enhancement in 1D



PDE examples [21]: Kuramoto-Sivashinsky model

$$\boxed{u_t + uu_x = -u_{xx} - u_{xxxx}.}$$

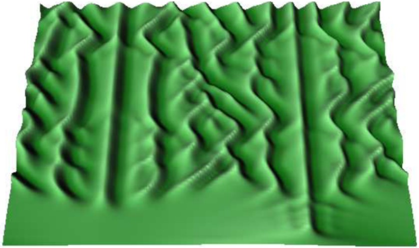


Fig. 2: Chaotic structure emerging from smooth initial data

PDE examples [22]: Burgers' equation

The simplest nonlinear example of a conservation law is the *inviscid Burgers equation*,

$$\boxed{u_t + (\tfrac{1}{2}u^2)_x \;=\; 0,} \tag{2}$$

i.e., $u_t + uu_x = 0$. This equation appears in studies of gas dynamics and traffic flow, and it serves as a prototype for nonlinear hyperbolic equations and conservation laws in general. It is the inviscid limit of the *Burgers equation* ($\rightarrow$ *ref*)

$$u_t + (\tfrac{1}{2}u^2)_x \;=\; \epsilon u_{xx}, \tag{3}$$

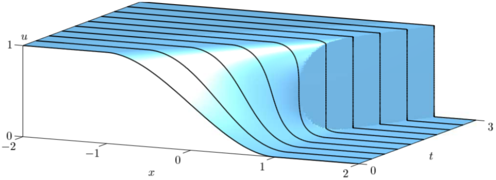


Fig. 1: Formation of a shock

PDE examples [23]: Ginzburg-Landau equations

$$\boxed{u_t = (1 + i\nu)u_{xx} + u - (1 + i\mu)u|u|^2, \qquad u \in \mathbb{C}}$$

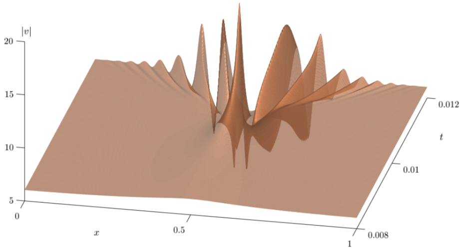


Fig. 2: Burst and collapse for a quintic complex Ginzburg–Landau equation

PDE examples [24]: Klein-Gordon model

$$\boxed{u_{tt} = \nabla^2 u - u,}$$

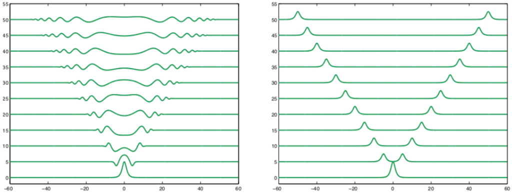
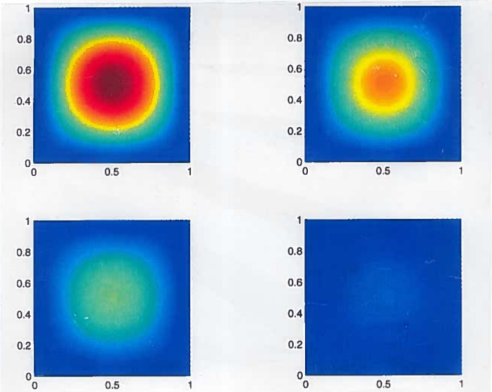


Fig. 1: Klein–Gordon (left) and wave equations (right)

PDEs in two space dimensions [1]



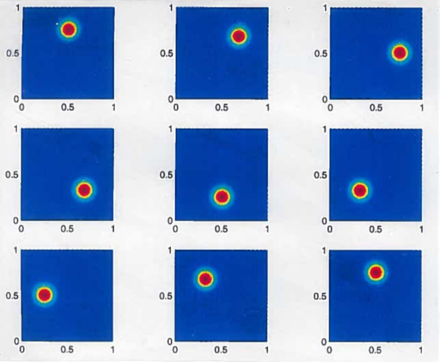
$$\boxed{\frac{\partial u}{\partial t} = \kappa . \Delta u}$$

- parabolic PDE
- heat equation
- solution damps out

PDEs in two space dimensions [2]

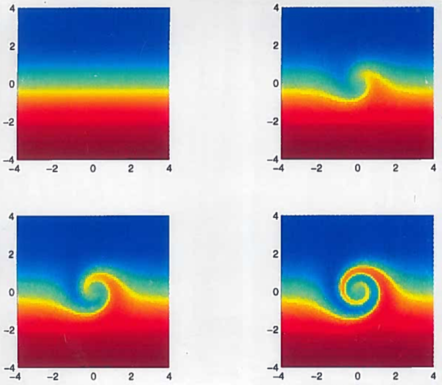
- solution = rotating cone (non-changing shape $\forall t \geq 0$)

$$\nabla \begin{cases} \text{heat equation with source (parabolic)}: & u_t = \Delta u + f \\ \text{transport equation (hyperbolic)}: & u_t + \beta_1 u_x + \beta_2 u_y = 0 \end{cases}$$



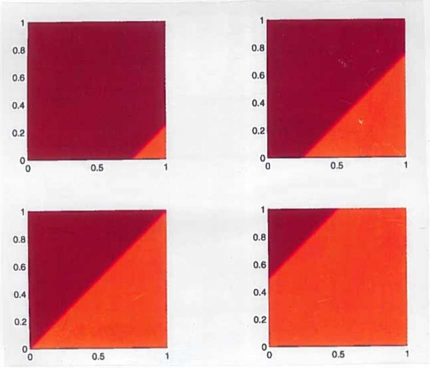
PDEs in two space dimensions [3]

- hyperbolic PDE
- hurricane model from meteorology
- developing 'whirlpool'



$$\boxed{\begin{gathered}\frac{\partial u}{\partial t} + \gamma_1 \frac{\partial u}{\partial x} + \gamma_2 \frac{\partial u}{\partial y} = 0 \\ \nabla \cdot \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = 0\end{gathered}}$$

PDEs in two space dimensions [4]



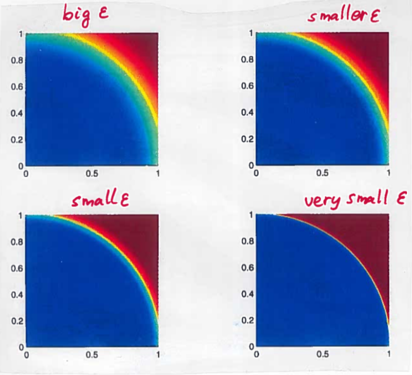
$$\boxed{\frac{\partial u}{\partial t} = \varepsilon \Delta u - u\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial y}}$$

small parameter

- hyperbolic PDE (nonlinear) — for $\varepsilon = 0$ & parabolic for $\varepsilon > 0$
- Burgers' equation
- travelling wave

PDEs in two space dimensions [5]

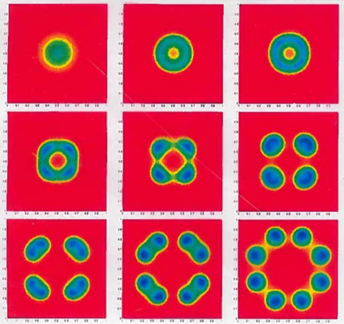
the same PDE, but with diffeent initial conditions and boundary conditions



solutions with steep transitions, depending on physical parameter in PDE model

PDEs in two space dimensions [6]

$$\boxed{\begin{aligned}\frac{\partial u}{\partial t} &= D_1\,\Delta u + f_1(u,v)\\ \frac{\partial v}{\partial t} &= D_2\,\Delta v + f_2(u,v)\end{aligned}}$$



- parabolic PDE (s)
- reaction-diffusion system from chemistry
- splitting pulses

Application areas

- weather prediction & climate models
- chemical reactions
- ecology, biology, ...
- traffic flow
- financial models
- geology, hydrology, ...
- languages, archaeology
- fluid flow, MHD
- water flows, rivers, oceans, ...
- image processing, visualization
- ETCETERA...!!!

Fourier series method [1]

Method of separation of variables

seek a solution of the form $u(x,t) = X(x)\,T(t)$ $\odot$

find

Example: heat equation in 1d

$$\boxed{u_t = \alpha^2 u_{xx} \quad , \quad x \in [0,L],\ t>0}$$

$$\boxed{u(0,t)=0 \quad , \quad u(L,t)=0 \quad , \quad t>0} \text{ BCs}$$

$$\boxed{u(x,0) = u_0(x),\ x \in [0,L]} \text{ IC}$$

$u \sim$ temperature
$L \sim$ length of bar
$u_0 \sim$ initial temperature

1) from $\odot$: $u_t = X(x).\dot{T}(t)$, $u_{xx} = X''(x)\,T(t)$

substitute $\Rightarrow$

$$\underbrace{\frac{\dot{T}(t)}{\alpha^2 T(t)}}_{\text{function of } t} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{function of } x}$$

must hold for all t and x under consideration $\Longrightarrow$

Fourier series method [2]

$$\Rightarrow \quad \frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = c \quad \leftarrow \text{constant} \qquad \Rightarrow \begin{cases} \dot{T}(t) - \alpha^2 c\, T(t) = 0 \\ X''(x) - c\, X(x) = 0 \end{cases}$$

2) apply BCs: $u(0,t) = X(0)T(t) = 0$ & $u(L,t) = X(L)T(t) = 0 \quad \forall t > 0$

if $T(t) = 0$ $\forall t>0$, then $u(x,t) = 0$ (trivial solution) $\quad$ or, if $X(0) = X(L) = 0$ (nontrivial solutions)

$$\Rightarrow \begin{cases} X''(x) - c\, X(x) = 0 \\ X(0) = X(L) = 0 \end{cases}$$

three cases

$$\begin{cases} c > 0,\ c = \lambda^2 \rightarrow X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x} \xrightarrow{BCs} C_1 = C_2 = 0 \rightarrow X = 0 \rightarrow u = 0 \ \times \\ c = 0 \rightarrow X(x) = C_3 + C_4 x \xrightarrow{BCs} C_3 = C_4 = 0 \rightarrow X = 0 \rightarrow u = 0 \ \times \\ c < 0,\ c = -\lambda^2 \rightarrow X(x) = C_5 \cos(\lambda x) + C_6 \sin(\lambda x) \xrightarrow{BCs} \begin{cases} C_5 = 0 \\ C_5 \cos(\lambda L) + C_6 \sin(\lambda L) = 0 \end{cases} \end{cases}$$

$$\underset{C_6 \neq 0}{\Rightarrow} \quad \sin(\lambda L) = 0 \ \Rightarrow \ \lambda L = n\pi, \ n = 0, \pm 1, \pm 2, \ldots$$

$$\lambda_n = \frac{n\pi}{L}, \ n = 1, 2, 3, \ldots \qquad (n = 0 \Rightarrow u = 0 \ \times, \ n = -1, -2, \ldots \Downarrow \text{ no extra information})$$

$$X_n(x) = a_n \sin\left(\frac{n\pi x}{L}\right), \ n = 1, 2, 3, \ldots$$

$\updownarrow$

"C_6"

Fourier series method [3]

3) apply IC : $c = -\lambda_n^2 = \left(\frac{n\pi}{L}\right)^2 \Rightarrow T_n(t) = b_n\, e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}$ $\quad n = 1, 2, 3, \dots$

$$\dot{T}(t) - \alpha^2 c\, T(t) = 0$$

$$\Rightarrow u_n(x,t) = X_n(x)\, T_n(t) = a_n \sin\left(\frac{n\pi x}{L}\right) \cdot b_n\, e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}$$

$$= c_n \cdot e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \cdot \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \dots$$

linear, homogeneous PDE $\Longrightarrow$ "superposition principle" $\quad u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n\, e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \cdot \sin\left(\frac{n\pi x}{L}\right)$

IC $\quad u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = u_0(x) \quad , \quad x \in [0, L]$

Fourier sine series expansion $= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow c_n = \frac{2}{L} \int_0^L u_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

we have solved the PDE

$$u(x,t) \approx \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{L}\right) \;\; \dots\dots\dots \; (?)$$

Fourier transform method [1]

$$\boxed{\begin{cases} u_t = \alpha^2 u_{xx} \\ u(x,0) = u_0(x) \end{cases}} \begin{array}{l} , x \in (-\infty, \infty), t > 0 \\ , x \in (-\infty, \infty) \quad \text{with } u \text{ and } u_x \to 0 \text{ as } x \to \pm\infty, t>0 \end{array}$$

this holds when $u_0(x)$ is continuous and $\int_{-\infty}^{\infty} |u_0(x)|\,dx < \infty$

1) transform PDE to ODE

apply Fourier transform to PDE and IC

$$\Rightarrow \mathcal{F}[u] = \hat{u}(\omega,t) \quad , \mathcal{F}[u_0(x)] = \hat{u}_0(\omega)$$

property $\mathcal{F}$-transform

$$\begin{cases} \mathcal{F}[u_t] = \mathcal{F}[\alpha^2 u_{xx}] \\ \mathcal{F}[u(x,0)] = \mathcal{F}[u_0(x)] \end{cases} \Rightarrow \begin{cases} \frac{d}{dt}\hat{u}(\omega,t) = -\alpha^2\omega^2\hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \hat{u}_0(\omega) \end{cases}$$

2) solve ODE : $\hat{u}(\omega,t) = \hat{u}_0(\omega).\, e^{-\alpha^2\omega^2 t}$

3) apply inverse transform (t fixed) : $u(x,t) = \mathcal{F}^{-1}[\hat{u}(\omega,t)] = \mathcal{F}^{-1}[\hat{u}_0(\omega)\, e^{-\alpha^2\omega t}]$

4) use "convolution property" : $u(x,t) = \mathcal{F}^{-1}[\hat{u}_0(\omega)] * \mathcal{F}^{-1}[e^{-\alpha^2\omega t}]$

$= \ldots$

Fourier transform method [2]

$$= u_0(x) * \left[\frac{1}{\sqrt{2\alpha^2 t}} e^{-\frac{x^2}{4\alpha^2 t}}\right]$$

property $\mathcal{F}$-transform

$$= \frac{1}{2\sqrt{\alpha^2 \pi t}} \int_{-\infty}^{\infty} u_0(\bar{x}) e^{-\frac{(x-\bar{x})^2}{4\alpha^2 t}} d\bar{x}$$

$$u(x,t) \approx \frac{1}{2\sqrt{\alpha^2 \pi t}} \int_{-M}^{M} u_0(\bar{x}) e^{-\frac{(x-\bar{x})^2}{4\alpha^2 t}} d\bar{x} \approx \text{----} \qquad (M = ?)$$

numerical quadrature

Another example: $\boxed{u_t + c u_x = 0}$ $\boxed{*}$ $u(x,0) = u_0(x)$

$$\mathcal{F}[u] = \hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx$$

$$\mathcal{F}^{-1}[\hat{u}] = u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega,t) e^{i\omega x} d\omega$$

apply to $\boxed{*}$ $\Longrightarrow$ $\begin{cases} \frac{d}{dt}\hat{u}(\omega,t) + c i \omega \hat{u}(\omega,t) = 0 & \text{(ODE)} \\ \hat{u}(\omega,0) = \hat{u}_0(\omega) \end{cases}$

Fourier transform method [3]

solve ODE $\Rightarrow$ $\hat{u}(\omega,t) = \hat{u}_0(\omega)\, e^{-i\omega c t}$

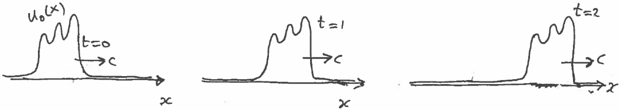
inverse F-transform $\Rightarrow$ $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega c t}\, \hat{u}_0(\omega)\, e^{i\omega x}\, d\omega$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_0(\omega)\, e^{i\omega(x-ct)}\, d\omega$$

$$= u_0(x-ct)$$

Note: $|\hat{u}(\omega,t)| = |\hat{u}_0(\omega)| \quad \forall t \geq 0$

" each Fourier component maintains its original amplitude and is modified only in phase"
(travelling wave behaviour)



Outlook to Lecture 2

- exercises for Lecture 1 (see webpage!)
- finite difference approximations of derivatives
- method of undetermined coefficients
- finite difference *matrices* and their *eigenvalues*
- treatment of boundary conditions
- non-uniform grids & transformations
- boundary-value models (stationary)