Lecture 1 E (xtra)

# (A) Classification of Second-Order PDEs

Classification of PDEs is an important concept because the general theory and methods of solution usually apply only to a given class of equations. Let us first discuss the classification of PDEs involving two independent variables.

## 1 Classification with two independent variables

Consider the following general second order linear PDE in two independent variables:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0, \tag{1}$$

where $A$, $B$, $C$, $D$, $E$, $F$ and $G$ are functions of the independent variables $x$ and $y$. The equation (1) may be written in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + f(x, y, u_x, u_y, u) = 0, \tag{2}$$

where

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}.$$

Assume that $A$, $B$ and $C$ are continuous functions of $x$ and $y$ possessing continuous partial derivatives of as high order as necessary.

The classification of PDE is motivated by the classification of second order algebraic equations in two-variables

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \tag{3}$$

We know that the nature of the curves will be decided by the principal part $ax^2+bxy+cy^2$ i.e., the term containing highest degree. Depending on the sign of the discriminant $b^2-4ac$, we classify the curve as follows:

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| If $b^2 - 4ac > 0$ then the curve traces hyperbola. |
| If $b^2 - 4ac = 0$ then the curve traces parabola. |
| If $b^2 - 4ac < 0$ then the curve traces ellipse. |

With suitable transformation, we can transform (3) into the following normal form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{(hyperbola)}.$$

$$x^2 = y \quad \text{(parabola)}.$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(ellipse)}.$$

**Linear PDE with constant coefficients.** Let us first consider the following general linear second order PDE in two independent variables $x$ and $y$ with constant coefficients:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0, \tag{4}$$

where the coefficients $A, B, C, D, E, F$ and $G$ are constants. The nature of the equation (4) is determined by the principal part containing highest partial derivatives i.e.,

$$Lu \equiv Au_{xx} + Bu_{xy} + Cu_{yy}. \tag{5}$$

For classification, we attach a symbol to (5) as $P(x, y) = Ax^2 + Bxy + Cy^2$ (as if we have replaced $x$ by $\frac{\partial}{\partial x}$ and $y$ by $\frac{\partial}{\partial y}$). Now depending on the sign of the discriminant $(B^2 - 4AC)$, the classification of (4) is done as follows:

$$B^2 - 4AC > 0 \Longrightarrow \text{Eq. (4) is hyperbolic} \tag{6}$$
$$B^2 - 4AC = 0 \Longrightarrow \text{Eq. (4) is parabolic} \tag{7}$$
$$B^2 - 4AC < 0 \Longrightarrow \text{Eq. (4) is elliptic} \tag{8}$$

**Linear PDE with variable coefficients.** The above classification of (4) is still valid if the coefficients $A, B, C, D, E$ and $F$ depend on $x$, $y$. In this case, the conditions (6), (7) and (8) should be satisfied at each point $(x, y)$ in the region where we want to describe its nature e.g., for elliptic we need to verify

$$B^2(x, y) - 4A(x, y)C(x, y) < 0$$

for each $(x, y)$ in the region of interest. Thus, we classify linear PDE with variable coefficients as follows:

$$B^2(x, y) - 4A(x, y)C(x, y) > 0 \text{ at } (x, y) \Longrightarrow \text{Eq. (4) is hyperbolic at } (x, y) \tag{9}$$
$$B^2(x, y) - 4A(x, y)C(x, y) = 0 \text{ at } (x, y) \Longrightarrow \text{Eq. (4) is parabolic at } (x, y) \tag{10}$$
$$B^2(x, y) - 4A(x, y)C(x, y) < 0 \text{ at } (x, y) \Longrightarrow \text{Eq. (4) is elliptic at } (x, y) \tag{11}$$

**Note:** Eq. (4) is hyperbolic, parabolic, or elliptic depends only on the coefficients of the second derivatives. It has nothing to do with the first-derivative terms, the term in $u$, or the nonhomogeneous term.

**EXAMPLE 1.**

1. $u_{xx} + u_{yy} = 0$ (Laplace equation). Here, $A = 1$, $B = 0$, $C = 1$ and $B^2 - 4AC = -4 < 0$. Therefore, it is an elliptic type.

2. $u_t = u_{xx}$ (Heat equation). Here, $A = -1$, $B = 0$, $C = 0$. $B^2 - 4AC = 0$. Thus, it is of parabolic type.

3. $u_{tt} - u_{xx} = 0$ (Wave equation). In this case, $A = -1$, $B = 0$, $C = 1$ and $B^2 - 4AC = 4 > 0$. Hence, it is of hyperbolic type.

4. $u_{xx} + xu_{yy} = 0, \quad x \neq 0$ (Tricomi equation). $B^2 - 4AC = -4x$. Given PDE is hyperbolic for $x < 0$ and elliptic for $x > 0$. This example shows that equations with variable coefficients can change form in the different regions of the domain.

## 2 Classification with more than two variables

Consider the second-order PDE in general form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = 0, \tag{12}$$

where the coefficients $a_{ij}, b_i, c$ and $d$ are functions of $x = (x_1, x_2, \cdots, x_n)$ alone and $u = u(x_1, x_2, \cdots, x_n)$.

Its principal part is

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{13}$$

It is enough to assume that $A = [a_{ij}]$ is symmetric if not, let $\bar{a}_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ and rewrite

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} \bar{a}_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{14}$$

Note that $\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$. As in two-space dimension, let us attach a quadratic form $P$ with (14) (i.e., replacing $\frac{\partial u}{\partial x_i}$ by $x_i$).

$$P(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j. \tag{15}$$

Since $A$ is a real valued symmetric ($a_{ij} = a_{ji}$) matrix, it is diagonalizable with real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (counted with their multiplicities). In other words, there exists a corresponding set of orthonormal set of $n$ eigenvectors, say $\sigma_1, \sigma_2, \cdots, \sigma_n$ with $R =$

$[\sigma_1, \sigma_2, \cdots, \sigma_n]$ as column vectors such that

$$R^T A R = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & \bigcirc & \\ & & \cdot & & \\ & & & \cdot & \\ & \bigcirc & & \cdot & \\ & & & & \lambda_n \end{bmatrix} = D \tag{16}$$

We now classify (12) depending on sign of eigenvalues of $A$:

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| (a) If $\lambda_i > 0\ \forall i$ or $\lambda_i < 0\ \forall i$ then (12) is elliptic type. |
| (b) If one or more of the $\lambda_i = 0$ then (12) is parabolic type. |
| (c) If one of the $\lambda_i < 0$ or $\lambda_i > 0$, and all the remaining have opposite sign then (12) is said to be of hyperbolic type. |

**Example 2.**

1. $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$. In this case, $\lambda_i = 1 > 0$ for all $i = 1, 2, 3$. It is an elliptic PDE since all eigenvalues are of one sign.
2. It is an easy exercise to check that $u_t - \nabla^2 u = 0$ is of parabolic type.
3. The equation $u_{tt} - \nabla^2 u = 0$ is of hyperbolic type.

**Example 3.** *Classify* $u_{x_1x_1} + 2(1 + cx_2)u_{x_2x_3} = 0$.

To symmetrize, write it as

$$u_{x_1x_1} + (1 + cx_2)u_{x_2x_3} + (1 + cx_2)u_{x_3x_2} = 0$$

i.e., $\partial_x^T A \partial_x - c\partial_{x_3} = 0$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 + cx_2 \\ 0 & 1 + cx_2 & 0 \end{bmatrix} \quad \partial_x = \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{bmatrix}$$

Eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1 + cx_2$, $\lambda_3 = -(1 + cx_2)$ and normalized eigenvectors

$$\sigma_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

So

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Note that $R = R^T = R^{-1}$.

$$R^T A R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + cx_2 & 0 \\ 0 & 0 & -(1 + cx_2) \end{bmatrix} = D$$

Equation is parabolic if $x_2 = -\frac{1}{c}$ $(c \neq 0)$, hyperbolic if $x_2 > -\frac{1}{c}$ and $x_2 < -\frac{1}{c}$. For $c = 0$, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$, it is hyperbolic type.

(B) ## Linear First-Order PDEs

The most general first-order linear PDE has the form

$$a(x,y)z_x + b(x,y)z_y + c(x,y)z = d(x,y), \qquad (1)$$

where $a$, $b$, $c$, and $d$ are given functions of $x$ and $y$. These functions are assumed to be continuously differentiable. Rewriting (1) as

$$a(x,y)z_x + b(x,y)z_y = -c(x,y)z + d(x,y), \qquad (2)$$

we observe that the left hand side of (2), i.e.,

$$a(x,y)z_x + b(x,y)z_y = \nabla z \cdot (a,b)$$

is (essentially) a directional derivative of $z(x,y)$ in the direction of the vector $(a,b)$, where $(a,b)$ is defined and nonzero. When $a$ and $b$ are constants, the vector $(a,b)$ had a fixed direction and magnitude, but now the vector can change as its base point $(x,y)$ varies. Thus, $(a,b)$ is a vector field on the plane.

The equations

$$\boxed{\frac{dx}{dt} = a(x,y), \qquad \frac{dy}{dt} = b(x,y),} \qquad (3)$$

determine a family of curves $x = x(t)$, $y = y(t)$ whose tangent vector $(\frac{dx}{dt}, \frac{dy}{dt})$ coincides with the direction of the vector $(a,b)$. Therefore, the derivative of $z(x,y)$ along these curves becomes

$$\begin{aligned}
\frac{dz}{dt} = \frac{d}{dt}z\{(x(t),y(t))\} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= z_x(x(t),y(t))a(x(t),y(t)) + z_y(x(t),y(t))b(x(t),y(t)) \\
&= -c(x(t),y(t))z(x(t),y(t)) + d(x(t),y(t)) \\
&= -c(t)z(t) + d(t),
\end{aligned}$$

where we have used the chain rule and (1). Thus, along these curves, $z(t) = z(x(t),y(t))$ satisfies the ODE

$$z'(t) + c(t)z(t) = d(t). \qquad (4)$$

Let $\mu(t) = \exp\left[\int_0^t c(\tau)d\tau\right]$ be an integrating factor for (4). Then, the solution is given by

$$z(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(\tau)d(\tau)d\tau + z(0)\right]. \qquad (5)$$

The approach described above to solve (1) by using the solutions of (3)-(4) is called **the method of characteristics**. It is based on the geometric interpretation of the partial differential equation (1).

**NOTE:** ($i$) The ODEs (3) is known as the characteristics equation for the PDE (1). The solution curves of the characteristic equation are the characteristics curves for (1).

($ii$) Observe that $\mu(t)$ and $d(t)$ depend only on the values of $c(x,y)$ and $d(x,y)$ along the characteristics curve $x = x(t)$, $y = y(t)$. Thus, equation (5) shows that the values $z(t)$ of the solution $z$ along the entire characteristics curve are completely determined, once the value $z(0) = z(x(0), y(0))$ is prescribed.

($iii$) Assuming certain smoothness conditions on the functions $a$, $b$, $c$, and $d$, the existence and uniqueness theory for ODEs guarantees a unique solution curve $(x(t), y(t), z(t))$ of (3)-(4) (i.e., a characteristic curve) passes through a given point $(x_0, y_0, z_0)$ in $(x, y, z)$-space.

## 1 The method of characteristics for solving linear first-order IVP

In practice we are not interested in determining a general solution of the partial differential equation (1) but rather a specific solution $z = z(x, y)$ that passes through or contains a given curve $C$. This problem is known as the initial value problem for (1). The method of characteristics for solving the initial value problem for (1) proceeds as follows.

Let the initial curve $C$ be given parametrically as:

$$x = x(s), \quad y = y(s), \quad z = z(s). \tag{6}$$

for a given range of values of the parameter $s$. The curve may be of finite or infinite extent and is required to have a continuous tangent vector at each point.

Every value of $s$ fixes a point on $C$ through which a unique characteristic curve passes (see, Fig. 2.1). The family of characteristic curves determined by the points of $C$ may be parameterized as

$$x = x(s,t), \quad y = y(s,t), \quad z = z(s,t)$$

with $t = 0$ corresponding to the initial curve $C$. That is, we have

$$x(s,0) = x(s), \quad y(s,0) = y(s), \quad z(s,0) = z(s).$$

In other words, we have the following:

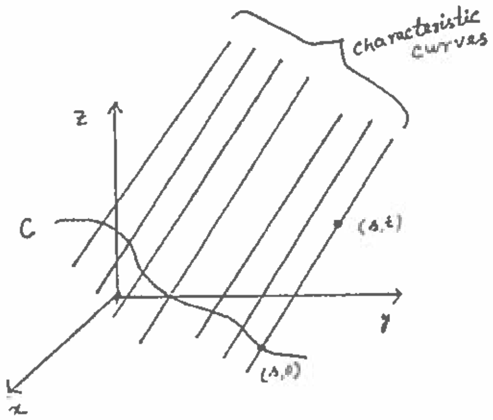


Figure 2.1: Characteristic curves and construction of the integral surface

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| The functions $x(s,t)$ and $y(s,t)$ are the solutions of the characteristics system (for each fixed s) $\frac{d}{dt}x(s,t) = a(x(s,t),y(s,t)), \quad \frac{d}{dt}y(s,t) = b(x(s,t),y(s,t))$ (7) with given initial values $x(s,0)$ and $y(s,0)$. |

Suppose that

$$z(x(s,0),y(s,0)) = g(s), \tag{8}$$

where $g(s)$ is a given function. We obtain $z(x(s,t),y(s,t))$ as follows: Let

$$z(s,t) = z(x(s,t),y(s,t)),\ c(s,t) = c(x(s,t),y(s,t)),\ d(s,t) = d(x(s,t),y(s,t)) \tag{9}$$

and

$$\mu(s,t) = \exp\left[\int_0^t c(s,t)dt\right]. \tag{10}$$

Analogous to formula (5), for each fixed $s$, we obtain

$$\boxed{z(s,t) = \frac{1}{\mu(s,t)}\left[\int_0^t \mu(s,t)d(s,t)dt + g(s)\right].} \tag{11}$$

$z(s,t)$ is the value of $z$ at the point $(x(s,t),y(s,t))$. Thus, as $s$ and $t$ vary, the point $(x,y,z)$, in $xyz$-space, given by

$$\boxed{x = x(s,t), \quad y = y(s,t), \quad z = z(s,t),} \tag{12}$$

traces out the surface of the graph of the solution $z$ of the PDE (1) which meets the initial curve (8). The equations (12) constitute the parametric form of the solution of (1) satisfying the initial condition (8) [i.e., a surface in $(x, y, z)$-space that contains the initial curve ]

**NOTE:** If the Jacobian $J(s,t) = x_s y_t - x_t y_s \neq 0$, then the equations $x = x(s,t)$ and $y = y(s,t)$ can be inverted to give $s$ and $t$ as (smooth) functions of $x$ and $y$ i.e., $s = s(x,y)$ and $t = t(x,y)$. The resulting function $z = z(x,y) = z(s(x,y), t(x,y))$ satisfies the PDE (1) in a neighborhood of the curve $C$ (in view of (4) and the initial condition (6)) and is the unique solution of the IVP.

**Example 1.** *Determine the solution the following IVP:*

$$\frac{\partial z}{\partial y} + c\frac{\partial z}{\partial x} = 0, \quad z(x,0) = f(x),$$

*where $f(x)$ is a given function and $c$ is a constant.*

Solution. A step by step procedure for the finding solution is given below.

**Step 1.**(Finding characteristic curves)

To apply the method of characteristics, parameterize the initial curve $C$ as follows: as follows:

$$x = s, \quad y = 0, \quad z = f(s). \tag{13}$$

The family of characteristics curves $x((s,t), y(s,t))$ are determined by solving the ODEs

$$\frac{d}{dt}x(s,t) = c, \quad \frac{d}{dt}y(s,t) = 1$$

The solution of the system is

$$x(s,t) = ct + c_1(s) \quad \text{and} \quad y(s,t) = t + c_2(s).$$

**Step 2.** (Applying IC)

Using the initial conditions

$$x(s,0) = s, \quad y(s,0) = 0.$$

we find that

$$c_1(s) = s, \quad c_2(s) = 0,$$

and hence

$$x(s,t) = ct + s \quad \text{and} \quad y(s,t) = t.$$

**Step 3.** (Writing the parametric form of the solution)
Comparing with (1), we have $c(x, y) = 0$ and $d(x, y) = 0$. Therefore, using (10) and (11), we find that

$$d(s,t) = 0, \quad \mu(s,t) = 1.$$

Since $z(x(s,0), y(s,0)) = z(s,0) = g(s) = f(s)$, we obtain $z(s,t) = f(s)$. Thus, the parametric form of the solution of the problem is given by

$$x(s,t) = ct + s, \quad y(s,t) = t, \quad z(s,t) = f(s).$$

**Step 4.** (Expressing $z(s,t)$ in terms of $z(x,y)$) Expressing $s$ and $t$ as $s = s(x,y)$ and $t = t(x,y)$, we have

$$s = x - cy, \quad t = y.$$

We now write the solution in the explicit form as

$$z(x,y) = z(s(x,y), y(x,y)) = f(x - cy).$$

Clearly, if $f(x)$ is differentiable, the solution $z(x,y) = f(x - cy)$ satisfies given PDE as well as the initial condition.

**NOTE:** Example 1 characterizes unidirectional wave motion with velocity $c$. If we consider the initial function $z(x,0) = f(x)$ to represent a waveform, the solution $z(x,y) = f(x - cy)$ shows that a point $x$ for which $x - cy =$ constant, will always occupy the same position on the wave form. If $c > 0$, the entire initial wave form $f(x)$ moves to the right without changing its shape with speed $c$ (if $c < 0$, the direction of motion is reversed).

**EXAMPLE 2.** *Find the parametric form of the solution of the problem*

$$-yz_x + xz_y = 0$$

*with the condition given by*

$$z(s, s^2) = s^3, \quad (s > 0).$$

**Solution.** To find the solution, let's proceed as follows.

**Step 1.** (Finding characteristic curves)

The family of characteristics curves $(x(s,t), y(s,t))$ are determined by solving

$$\frac{d}{dt}x(s,t) = -y(s,t), \quad \frac{d}{dt}y(s,t) = x(s,t)$$

with initial conditions

$$x(s,0) = s, \quad y(s,0) = s^2.$$

The general solution of the system is

$$x(s,t) = c_1(s)\cos(t) + c_2(s)\sin(t) \quad \text{and} \quad y(s,t) = c_1(s)\sin(t) - c_2(s)\cos(t).$$

**Step 2.** (Applying IC)

Using ICs, we find that

$$c_1(s) = s, \quad c_2(s) = -s^2,$$

and hence

$$x(s,t) = s\cos(t) - s^2\sin(t) \quad \text{and} \quad y(s,t) = s\sin(t) + s^2\cos(t).$$

**Step 3.** (Writing the parametric form of the solution)

Comparing with (1), we note that $c(x,y) = 0$ and $d(x,y) = 0$. Therefore, using (10) and (11), it follows that

$$d(s,t) = 0, \quad \mu(s,t) = 1.$$

In view of the given condition curve and $z = z(s,t)$, we obtain

$$z(x(s,0), y(s,0)) = z(s, s^2) = g(s) = s^3, \quad z(s,t) = s^3.$$

Thus, the parametric form of the solution of the problem is given by

$$x(s,t) = s\cos(t) - s^2\sin(t), \quad y(s,t) = s\sin(t) + s^2\cos(t), \quad z(s,t) = s^3.$$

**Step 4.** (Expressing $z(s,t)$ in terms of $z(x,y)$)

Writing $s$ and $t$ as a function of $x$ and $y$, it is an easy exercise to show that

$$z(x,y) = \frac{1}{\sqrt{8}}\left[-1 + \sqrt{1 + 4(x^2 + y^2)}\right]^{3/2}.$$

## Ⓒ Canonical Form

By a suitable change of the independent variables we shall show that any equation of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0, \quad (1)$$

where $A$, $B$, $C$, $D$, $E$, $F$ and $G$ are functions of the variables $x$ and $y$, can be reduced to a canonical form or normal form. The transformed equation assumes a simple form so that the subsequent analysis of solving the equation will be become easy.

Consider the transformation of the indpendent variables from $(x, y)$ to $(\xi, \eta)$ given by

$$\xi = \xi(x, y), \quad \eta = \eta(x, y). \quad (2)$$

Here, the functions $\xi$ and $\eta$ are continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = (\xi_x\eta_y - \xi_y\eta_x) \neq 0 \quad (3)$$

in the domain where (1) holds.

Using chain rule, we notice that

$$\begin{aligned}
u_x &= u_\xi\xi_x + u_\eta\eta_x \\
u_y &= u_\xi\xi_y + u_\eta\eta_y \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy} \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}
\end{aligned}$$

Substituting these expression into (1), we obtain

$$\bar{A}(\xi_x, \xi_y)u_{\xi\xi} + \bar{B}(\xi_x, \xi_y; \eta_x, \eta_y)u_{\xi\eta} + \bar{C}(\eta_x, \eta_y)u_{\eta\eta} = F(\xi, \eta, u(\xi, \eta), u_\xi(\xi, \eta), u_\eta(\xi, \eta)), \quad (4)$$

where

$$\begin{aligned}
\bar{A}(\xi_x, \xi_y) &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
\bar{B}(\xi_x, \xi_y; \eta_x, \eta_y) &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
\bar{C}(\eta_x, \eta_y) &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2.
\end{aligned}$$

An easy calculation shows that

$$\bar{B}^2 - 4\bar{A}\bar{C} = (\xi_x\eta_y - \xi_y\eta_x)^2(B^2 - 4AC). \quad (5)$$

The equation (5) shows that the transformation of the independent variables does not modify the type of PDE.

We shall determine $\xi$ and $\eta$ so that (4) takes the simplest possible form. We now consider the following cases:

**Case I**: $B^2 - 4AC > 0$ (Hyperbolic type)
**Case II**: $B^2 - 4AC = 0$ (Parabolic type)
**Case III**: $B^2 - 4AC < 0$ (Elliptic type)

**Case I**: Note that $B^2 - 4AC > 0$ implies the equation $A\alpha^2 + B\alpha + C = 0$ has two real and distinct roots, say $\lambda_1$ and $\lambda_2$. Now, choose $\xi$ and $\eta$ such that

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y} \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y}. \tag{6}$$

Then the coefficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ will be zero because

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = (A\lambda_1^2 + B\lambda_1 + C)\xi_y^2 = 0,$$
$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = (A\lambda_2^2 + B\lambda_2 + C)\eta_y^2 = 0.$$

Thus, (5) reduces to

$$\bar{B}^2 = (B^2 - AC)(\xi_x\eta_y - \xi_y\eta_x)^2 > 0$$

as $B^2 - 4AC > 0$. Note that (6) is a first-order linear PDE in $\xi$ and $\eta$ whose characteristics curves are satisfy the first-order ODEs

$$\frac{dy}{dx} + \lambda_i(x, y) = 0, \quad i = 1, 2. \tag{7}$$

Let the family of curves determined by the solution of (7) for $i = 1$ and $i = 2$ be

$$f_1(x, y) = c_1 \quad \text{and} \quad f_2(x, y) = c_2, \tag{8}$$

respectively. These family of curves are called characteristics curves of PDE (5). With this choice, divide (4) throughout by $\bar{B}$ (as $\bar{B} > 0$) and use (7)-(8) to obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \phi(\xi, \eta, u, u_\xi, u_\eta), \tag{9}$$

which is the canonical form of hyperbolic equation.

**Example 1.** *Reduce the equation $u_{xx} = x^2 u_{yy}$ to its canonical form.*

**Solution.** Comparing with (1) we find that $A = 1$, $B = 0$, $C = -x^2$.

The roots of the equations $A\alpha^2 + B\alpha + C = 0$ i.e., $\alpha^2 + x^2 = 0$ are given by $\lambda_i = \pm x$. The differential equations for the family of characteristics curves are

$$\frac{dy}{dx} \pm x = 0.$$

whose solutions are $y + \frac{1}{2}x^2 = c_1$ and $y - \frac{1}{2}x^2 = c_2$. Choose

$$\xi = y + \frac{1}{2}x^2, \quad \eta = y - \frac{1}{2}x^2.$$

An easy computation shows that

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x, \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
&= u_{\xi\xi}x^2 - 2u_{\xi\eta}x^2 + u_{\eta\eta}x^2 + u_\xi - u_\eta, \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}, \\
&= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.
\end{aligned}$$

Substituting these expression in the equation $u_{xx} = x^2 u_{yy}$ yields

$$4x^2 u_{\xi\eta} = (u_\xi - u_\eta)$$

$$\text{or} \quad 4(\xi - \eta)u_{\xi\eta} = \frac{1}{4(\xi - \eta)}(u_\xi - u_\eta)$$

$$\text{or} \quad u_{\xi\eta} = \frac{1}{4(\xi - \eta)}(u_\xi - u_\eta)$$

which is the required canonical form.

**CASE II**: $B^2 - 4AC = 0 \implies$ the equation $A\alpha^2 + B\alpha + C = 0$ has two equal roots, say $\lambda_1 = \lambda_2 = \lambda$. Let $f_1(x, y) = c_1$ be the solution of $\frac{dy}{dx} + \lambda(x, y) = 0$. Take $\xi = f_1(x, y)$ and $\eta$ to be the any function of $x$ and $y$ which is independent of $\xi$.

As before, $\bar{A}(\xi_x, \xi_y) = 0$ and hence from equation (5), we obtain $\bar{B} = 0$. Note that $\bar{C}(\eta_x, \eta_y) \neq 0$, otherwise $\eta$ would be a function of $\xi$. Dividing (4) by $\bar{C}$, the canonical form of (2) is

$$u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta). \qquad (10)$$

which is the canonical form of parabolic equation.

**EXAMPLE 2.** *Reduce the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$ to canonical form.*

**Solution.** In this case, $A = 1$, $B = 2$, $C = 1$. The equation $\alpha^2 + 2\alpha + 1 = 0$ has equal roots $\lambda = -1$. The solution of $\frac{dy}{dx} - 1 = 0$ is $x - y = c_1$ Take $\xi = x - y$. Choose $\eta = x + y$. proceed as in Example 1 to obtain $u_{\eta\eta} = 0$ which is the canonical form of the given PDE.

**CASE III**: When $B^2 - 4AC < 0$, the roots of $A\alpha^2 + B\alpha + C = 0$ are complex. Following the procedure as in CASE I, we find that

$$u_{\xi\eta} = \phi_1(\xi, \eta, u, u_\xi, u_\eta). \qquad (11)$$

The variables $\xi$, $\eta$ are infact complex conjugates. To get a real canonical form use the transformation

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = \frac{1}{2i}(\xi - \eta)$$

to obtain

$$u_{\xi\eta} = \frac{1}{4}(u_{\alpha\alpha} + u_{\beta\beta}), \qquad (12)$$

which follows from the following calculation:

$$\begin{aligned}
u_\xi &= u_\alpha \alpha_\xi + u_\beta \beta_\xi = \frac{1}{2}u_\alpha + \frac{1}{2i}u_\beta \\
u_{\xi\eta} &= \frac{1}{2}(u_{\alpha\alpha}\alpha_\eta + u_{\alpha\beta}\beta_\eta) + \frac{1}{2i}(u_{\beta\alpha}\alpha_\eta + u_{\beta\beta}\beta_\eta) \\
&= \frac{1}{4}(u_{\alpha\alpha} + u_{\beta\beta}).
\end{aligned}$$

The desired canonical form is

$$u_{\alpha\alpha} + u_{\beta\beta} = \psi(\alpha, \beta, u(\alpha, \beta), u_\alpha(\alpha, \beta), u_\beta(\alpha, \beta)). \qquad (13)$$

**EXAMPLE 3.** *Reduce the equation $u_{xx} + x^2 u_{yy} = 0$ to canonical form.*

**Solution.** In this case, $A = 1$, $B = 0$, $C = x^2$. The roots are $\lambda_1 = ix$, $\lambda_2 = -ix$. Take $\xi = iy + \frac{1}{2}x^2$, $\eta = -iy + \frac{1}{2}x^2$. Then $\alpha = \frac{1}{2}x^2$, $\beta = y$...............

D

## Method of Separation of Variables

Separation of variables is one of the oldest technique for solving initial-boundary value problems (IBVP) and applies to problems, where

- PDE is linear and homogeneous (not necessarily constant coefficients) and
- BC are linear and homogeneous.

Basic Idea: To seek a solution of the form

$$u(x,t) = X(x)T(t),$$

where $X(x)$ is some function of $x$ and $T(t)$ in some function of $t$. The solutions are simple because any temperature $u(x,t)$ of this form will retain its basic "shape" for different values of time $t$. The separation of variables reduced the problem of solving the PDE to solving the two ODEs: One second order ODE involving the independent variable $x$ and one first order ODE involving $t$. These ODEs are then solved using given initial and boundary conditions.

To illustrate this method, let us apply to a specific problem. Consider the following IBVP:

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \quad 0 \le x \le L,\ 0 < t < \infty, \tag{1}$$

$$\text{BC:} \quad u(0,t) = 0 \ \ u(L,t) = 0, \quad 0 < t < \infty, \tag{2}$$

$$\text{IC:} \quad u(x,0) = f(x), \quad 0 \le x \le L. \tag{3}$$

**Step 1:**(Reducing to the ODEs) Assume that equation (1) has solutions of the form

$$\boxed{u(x,t) = X(x)T(t),}$$

where $X$ is a function of $x$ alone and $T$ is a function of $t$ alone. Note that

$$u_t = X(x)T'(t) \quad \text{and} \quad u_{xx} = X''(x)T(t).$$

Now, substituting these expression into $u_t = \alpha^2 u_{xx}$ and separating variables, we obtain

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

$$\Rightarrow \quad \frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}.$$

Since a function of $t$ can equal a function of $x$ only when both functions are constant. Thus,

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = c$$

for some constant $c$. This leads to the following two ODEs:

$$T'(t) - \alpha^2 c T(t) = 0, \tag{4}$$

$$X''(x) - cX(x) = 0. \tag{5}$$

Thus, the problem of solving the PDE (1) is now reduced to solving the two ODEs.

**Step 2**:(Applying BCs)

Since the product solutions $u(x,t) = X(x)T(t)$ are to satisfy the BC (2), we have

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad X(L)T(t) = 0, \quad t > 0.$$

Thus, either $T(t) = 0$ for all $t > 0$, which implies that $u(x,t) = 0$, or $X(0) = X(L) = 0$. Ignoring the trivial solution $u(x,t) = 0$, we combine the boundary conditions $X(0) = X(L) = 0$ with the differential equation for $X$ in (5) to obtain the BVP:

$$X''(x) - cX(x) = 0, \quad X(0) = X(L) = 0. \tag{6}$$

There are three cases: $c < 0$, $c > 0$, $c = 0$ which will be discussed below. It is convenient to set $c = -\lambda^2$ when $c < 0$ and $c = \lambda^2$ when $c > 0$, for some constant $\lambda > 0$.

*Case 1.* ($c = \lambda^2 > 0$ for some $\lambda > 0$). In this case, a general solution to the differential equation (5) is

$$X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x},$$

where $C_1$ and $C_2$ are arbitrary constants. To determine $C_1$ and $C_2$, we use the BC $X(0) = 0$, $X(L) = 0$ to have

$$X(0) = C_1 + C_2 = 0, \tag{7}$$

$$X(L) = C_1 e^{\lambda L} + C_2 e^{-\lambda L} = 0. \tag{8}$$

From the first equation, it follows that $C_2 = -C_1$. The second equation leads to

$$\begin{aligned} & C_1(e^{\lambda L} - e^{-\lambda L}) = 0, \\ \Rightarrow \quad & C_1(e^{2\lambda L} - 1) = 0, \\ \Rightarrow \quad & C_1 = 0. \end{aligned}$$

since $(e^{2\lambda L} - 1) > 0$ as $\lambda > 0$. Therefore, we have $C_1 = 0$ and hence $C_2 = 0$. Consequently $X(x) = 0$ and this implies $u(x,t) = 0$ i.e., there is no nontrivial solution to (5) for the case $c > 0$.

*Case 2.* (when c=0)
The general solution solution to (5) is given by

$$X(x) = C_3 + C_4 x.$$

Applying BC yields $C_3 = C_4 = 0$ and hence $X(x) = 0$. Again, $u(x,t) = X(x)T(t) = 0$. Thus, there is no nontrivial solution to (5) for $c = 0$.

*Case 3.* (When $c = -\lambda^2 < 0$ for some $\lambda > 0$)

The general solution to (5) is

$$X(x) = C_5 \cos(\lambda x) + C_6 \sin(\lambda x).$$

This time the BC $X(0) = 0,\ \ X(L) = 0$ gives the system

$$\begin{aligned} C_5 &= 0, \\ C_5 \cos(\lambda L) + C_6 \sin(\lambda L) &= 0. \end{aligned}$$

As $C_5 = 0$, the system reduces to solving $C_6 \sin(\lambda L) = 0$. Hence, either $\sin(\lambda L) = 0$ or $C_6 = 0$. Now

$$\sin(\lambda L) = 0 \implies \lambda L = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Therefore, (5) has a nontrivial solution ($C_6 \neq 0$) when

$$\lambda L = n\pi \quad \text{or} \quad \lambda = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots.$$

Here, we exclude $n = 0$, since it makes $c = 0$. Therefore, the nontrivial solutions (eigenfunctions) $X_n$ corresponding to the eigenvalue $c = -\lambda^2$ are given by

$$X_n(x) = a_n \sin(\frac{n\pi x}{L}), \tag{9}$$

where $a_n$'s are arbitrary constants.

**Step 3:**(Applying IC)

Let us consider solving equation (4). The general solution to (4) with $c = -\lambda^2 = (\frac{n\pi}{L})^2$ is

$$T_n(t) = b_n e^{-\alpha^2 (\frac{n\pi}{L})^2 t}.$$

Combing this with (9), the product solution $u(x,t) = X(x)T(t)$ becomes

$$\begin{aligned} u_n(x,t) &:= X_n(x)T_n(t) = a_n \sin(\frac{n\pi x}{L}) b_n e^{-\alpha^2(\frac{n\pi}{L})^2 t} \\ &= c_n e^{-\alpha^2(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi x}{L}), \quad n = 1, 2, 3, \ldots, \end{aligned}$$

where $c_n$ is an arbitrary constant.

Since the problem (9) is linear and homogeneous, an application of superposition principle gives

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi x}{L}),} \qquad (10)$$

which will be a solution to (1)-(3), provided the infinite series has the proper convergence behavior.

Since the solution (10) is to satisfy IC (3), we must have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x), \quad 0 < x < L.$$

Thus, if $f(x)$ has an expansion of the form

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right), \qquad (11)$$

which is called a Fourier sine series (FSS) with $c_n$'s are given by the formula

$$c_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx. \qquad (12)$$

Then the infinite series (10) with the coefficients $c_n$ given by (12) is a solution to the problem (1)-(3).

**Example 1.** *Find the solution to the following IBVP:*

$$\begin{aligned} u_t &= 3u_{xx} \quad 0 \le x \le \pi, \ 0 < t < \infty, && (13) \\ u(0,t) &= u(\pi,t) = 0, \quad 0 < t < \infty, && (14) \\ u(x,0) &= 3\sin 2x - 6\sin 5x, \quad 0 \le x \le \pi. && (15) \end{aligned}$$

**Solution.** Comparing (13) with (1), we notice that $\alpha^2 = 3$ and $L = \pi$. Using formula (10), we write a solution $u(x,t)$ as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-3n^2 t} \sin(nx).$$

To determine $c_n$'s, we use IC (15) to have

$$u(x,0) = 3\sin 2x - 6\sin 5x = \sum_{n=1}^{\infty} c_n \sin(nx).$$

Comparing the coefficients of like terms, we obtain

$$c_2 = 3 \quad \text{and} \quad c_5 = -6,$$

and the remaining $c_n$'s are zero. Hence, the solution to the problem (13)-(15) is

$$\begin{aligned} u(x,t) &= c_2 e^{-3(2)^2 t} \sin(2x) + c_5 e^{-3(5)^2 t} \sin(5x) \\ &= 3e^{-12t} \sin(2x) - 6e^{-75t} \sin(5x). \end{aligned}$$

(E)

We shall study some applications of the Fourier transform in solving the heat flow problems where the spatial domain is infinite or semi-infinite.

## 1 Heat flow problem in an infinite rod

Consider the heat flow in an infinite rod where the initial temperature is $u(x,0) = f(x)$. We shall prove that if the function $f(x)$ is continuous and either absolutely integrable i.e.,

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$$

or bounded (i.e., $|f(x)| \le M\ \forall x$), then the following IVP problem has a solution $u(x,t)$ which is continuous throughout the half-plane $t \ge 0$, $-\infty < x < \infty$.

$$\text{PDE:} \quad u_t(x,t) = \alpha^2 u_{xx}(x,t), \quad -\infty < x < \infty,\ t > 0, \tag{1}$$

$$\text{IC:} \quad u(x,0) = f(x), \quad -\infty < x < \infty, \tag{2}$$

with $u(x,t)$, $u_x(x,t) \to 0$ as $x \to \pm\infty$, $t > 0$.

The stepwise solution procedure is given below.

**Step 1.** (*Transforming the problem to an IVP in ODE*)
We apply FT $\mathcal{F}$ to the PDE (1) and IC (2) and use the properties of FT to reduce the given Cauchy problem to an IVP for an ODE. Let

$$\mathcal{F}[u] = \hat{u}(\omega,t) \quad \mathcal{F}[f(x)] = \hat{f}(\omega).$$

Taking the FT of both sides of the PDE (1) and IC (2) with respect to the $x$ variable, we obtain

$$\mathcal{F}[u_t] = \alpha^2 \mathcal{F}[u_{xx}]$$
$$\mathcal{F}[u(x,0)] = \mathcal{F}[f(x)].$$

Using the properties of the FT

$$\mathcal{F}[u_t] = \frac{d}{dt}\hat{u}(\omega,t), \quad \mathcal{F}[u_{xx}] = -\omega^2 \hat{u}(\omega,t)$$

we have

$$\frac{d}{dt}\hat{u}(\omega,t) = -\alpha^2\omega^2\hat{u}(\omega,t), \tag{3}$$

$$\hat{u}(\omega,0) = \hat{f}(\omega). \tag{4}$$

**Step 2.** (*Solving the transformed problem*)

Note that (3) is a first-order IVP for an ODE in $t$ for each fixed $\omega$. The solution to this problem is given by

$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-\alpha^2\omega^2 t}. \tag{5}$$

**Step 3.** (*Finding the inverse transform*)

To find the solution $u(x,t)$, we take inverse transform, with $t$ fixed, to obtain

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[\hat{u}(\omega,t)] \\ &= \mathcal{F}^{-1}[\hat{f}(\omega)e^{-\alpha^2\omega^2 t}]. \end{aligned}$$

**Step 4.** (*Using convolution property of the inverse FT*)

Using the convolution property of $\mathcal{F}^{-1}$, we write

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}[\hat{f}(\omega)e^{-\alpha^2\omega^2 t}] \\ &= \mathcal{F}^{-1}[\hat{f}(\omega)] * \mathcal{F}^{-1}[e^{-\alpha^2\omega^2 t}] \\ &= f(x) * \left[\frac{1}{\sqrt{2\alpha^2 t}} e^{-(\frac{x^2}{4\alpha^2 t})}\right] \\ &= \frac{1}{2\sqrt{\alpha^2\pi t}} \int_{-\infty}^{\infty} f(\omega) e^{-\frac{(x-\omega)^2}{4\alpha^2 t}}\, d\omega. \end{aligned}$$

REMARK 1.

- Note that integrand is made up of two terms i.e., the initial temperature $f(x)$ and the function
$$G(x,t) = \frac{1}{2\sqrt{\alpha^2\pi t}} e^{-\frac{(x-\omega)^2}{4\alpha^2 t}}.$$
The function $G(x,t)$ is called Green's function or impulse-response function which is the temperature response to an initial temperature impulse at $x = \omega$.

- The major drawback of the FT method is that all functions can not be transformed. Only functions that damp to zero sufficiently fast as $|x| \to \infty$ have FTs.

## 2 Heat flow problem in a semi-infinite rod

Consider the heat flow in a semi-infinite region with the temperature prescribed as a function of time at $x = 0$.

**Example 2.** *Solve the problem*

$$\text{PDE:} \quad u_t(x,t) = \alpha^2 u_{xx}(x,t), \quad 0 < x < \infty, \ t > 0 \tag{6}$$

$$\text{BC:} \quad u(0,t) = b_0 \quad t > 0, \tag{7}$$

$$\text{IC:} \quad u(x,0) = 0, \quad -\infty < x < \infty, \tag{8}$$

*with* $u(x,t)$, $u_x(x,t) \to 0$ *as* $x \to \infty$.

Since $0 < x < \infty$, we may wish to use a transform. Since u is specified at $x = 0$, we should try to use Fourier sine transform (and not the Fourier cosine transform). We solve this problem with the following steps.

**Step 1.** (*Transforming the problem*)
Notice that $u$ is specified at $x = 0$. Let $\mathcal{F}_s[u] = \hat{u}_s(\omega, t)$. Now taking FST of both sides of (6) and noting the following properties of FST

$$\begin{aligned}
\mathcal{F}_s[u_t] &= \sqrt{\frac{2}{\pi}} \int_0^\infty u_t(x,t)\sin(\omega x)dx \\
&= \frac{d}{dt}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty u_t(x,t)\sin(\omega x)dx\right] \\
&= \frac{d}{dt}\mathcal{F}_s[u] \\
&= \frac{d}{dt}\hat{u}_s(\omega,t).
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_s[u_{xx}] &= -\omega^2 \mathcal{F}_s[u] + \sqrt{\frac{2}{\pi}}\omega u(0,t) \\
&= -\omega^2 \hat{u}_s(\omega,t) + \sqrt{\frac{2}{\pi}}\omega u(0,t) \\
&= -\omega^2 \hat{u}_s(\omega,t) + \sqrt{\frac{2}{\pi}} b_0 \omega,
\end{aligned}$$

where in the last step we have used BC $u(0,t) = b_0$, we arrive at the ODE

$$\frac{d}{dt}\hat{u}_s(\omega,t) = \alpha^2\left(-\omega^2 \hat{u}_s(\omega,t) + \sqrt{\frac{2}{\pi}} b_0 \omega\right).$$

Next, taking FST of the IC (8), we obatin

$$\mathcal{F}_s[u(x,0)] = \mathcal{F}_s[0] \longrightarrow \hat{u}_s(\omega,0) = 0.$$

Thus, we transform the original problem (6)-(8) to an IVP in ODE:

$$\begin{aligned}
&\frac{d}{dt}\hat{u}_s(\omega,t) + \alpha^2\omega^2\hat{u}_s(\omega,t) = \sqrt{\frac{2}{\pi}}\alpha^2 b_0 \omega, \\
&\hat{u}_s(\omega,0) = 0.
\end{aligned}$$

**Step 2.**(*Solving the transformed problem*)

Using the standard method of solving ODE, the solution is given by

$$\hat{u}_s(\omega, t) = \sqrt{\frac{2}{\pi}}\frac{b_0}{\omega}(1 - e^{-\omega^2\alpha^2 t}). \tag{9}$$

**Step 3.** (*Finding the Inverse Transform*)

Applying the inverse FST to both sides of (9), we find that

$$\begin{aligned} u(x,t) = \mathcal{F}_s^{-1}[\hat{u}_s(\omega,t)] &= \mathcal{F}_s^{-1}\left[\sqrt{\frac{2}{\pi}}\frac{b_0}{\omega}(1 - e^{-\omega^2\alpha^2 t})\right] \\ &= \frac{2}{\pi}b_0 \int_0^\infty \frac{\sin(\omega x)}{\omega}(1 - e^{-\alpha^2\omega^2 t})d\omega \\ &= b_0\left[erfc(\frac{x}{\sqrt{2\alpha^2 t}})\right], \end{aligned}$$

where $erfc(y)$ is the complementary error function given by

$$erfc(y) = \sqrt{\frac{2}{\pi}}\int_y^\infty e^{-\tau^2}d\tau.$$

Hence, the solution of the heat conduction problem is

$$u(x,t) = b_0\, erfc\left(\frac{x}{\sqrt{2\alpha^2 t}}\right).$$