

Lecture 2

Paul Andries Zegeling

Department of Mathematics, Utrecht University

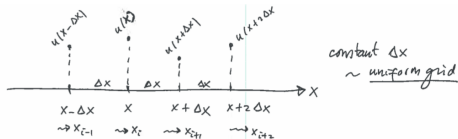
Numerical Methods for PDEs

Outline of Lecture 2

- ⌈ exercises of Lecture 1
- ⌋ finite difference approximations of derivatives
- ⌈ method of undetermined coefficients
- ∇ finite difference *matrices* and their *eigenvalues*
- ⌈ non-uniform grids & transformations
- ⊕ boundary-value models (stationary) ⇒ exercises!
- ⌋ outlook to Lecture 3

Finite differences [1]

Calculus of finite differences:



forward difference: $\Delta_+ [u](x) = u(x+\Delta x) - u(x)$

forward difference approximation \rightarrow

$$D_{1+} u(x) = \frac{u(x+\Delta x) - u(x)}{\Delta x} \rightarrow \text{matrix } D_{1+} = \frac{1}{\Delta x} \begin{pmatrix} \ddots & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

diagonal

backward difference: $\Delta_- [u](x) = u(x) - u(x-\Delta x)$

backward difference approximation \rightarrow

$$D_{1-} u(x) = \frac{u(x) - u(x-\Delta x)}{\Delta x} \rightarrow \text{matrix } D_{1-} = \frac{1}{\Delta x} \begin{pmatrix} \ddots & & & \\ & 1 & & \\ & & -1 & \\ & & & \ddots \end{pmatrix}$$

diagonal

central difference: $\Delta_c [u](x) = u(x+\frac{1}{2}\Delta x) - u(x-\frac{1}{2}\Delta x)$

central difference approximation \rightarrow

$$D_{1c} u(x) = \frac{u(x+\frac{1}{2}\Delta x) - u(x-\frac{1}{2}\Delta x)}{\Delta x} \rightarrow \text{matrix } D_{1c} = \frac{1}{2\Delta x} \begin{pmatrix} \ddots & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

diagonal

OR $\frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}$

Finite differences [2]

higher-order: central: $u''(x) \approx \frac{\Delta_c^2[u](x)}{(\Delta x)^2} = \frac{\frac{u(x+\Delta x) - u(x)}{\Delta x} - \frac{u(x) - u(x-\Delta x)}{\Delta x}}{\Delta x}$

$$= \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{(\Delta x)^2}$$

forward: $u''(x) \approx \frac{\Delta_{1+}^2[u](x)}{(\Delta x)^2} = \frac{u(x+2\Delta x) - u(x+\Delta x)}{\Delta x} - \frac{u(x+\Delta x) - u(x)}{\Delta x}$

$$= \frac{u(x+2\Delta x) - 2u(x+\Delta x) + u(x)}{(\Delta x)^2}$$

backward: $u''(x) \approx \frac{\Delta_{1-}^2[u](x)}{(\Delta x)^2} = \frac{u(x) - 2u(x-\Delta x) + u(x-2\Delta x)}{(\Delta x)^2}$

$$= \frac{u(x) - 2u(x-\Delta x) + u(x-2\Delta x)}{(\Delta x)^2}$$

↙ $\Delta_c(\Delta_c[u])(x) = \Delta_c[u](x+\frac{1}{2}\Delta x) - \Delta_c[u](x-\frac{1}{2}\Delta x)$

$$= (u(x+\Delta x) - u(x)) - (u(x) - u(x-\Delta x))$$

$$= u(x+\Delta x) - 2u(x) + u(x-\Delta x)$$

Finite differences [3]

In general: $\Delta_{1+}^m [u](x) = \sum_{i=0}^m (-1)^i \binom{m}{i} u(x + (m-i)\Delta x)$

$$\Delta_{1-}^m [u](x) = \sum_{i=0}^m (-1)^i \binom{m}{i} u(x - i\Delta x)$$

$$\Delta_{1c}^m [u](x) = \sum_{i=0}^m (-1)^i \binom{m}{i} u(x + (\frac{m}{2} - i)\Delta x)$$

$$u^{(m)}(x) = \frac{\Delta_{1+}^m [u](x)}{(\Delta x)^m} + \mathcal{O}(\Delta x) = \frac{\Delta_{1-}^m [u](x)}{(\Delta x)^m} + \mathcal{O}(\Delta x) = \frac{\Delta_{1c}^m [u](x)}{(\Delta x)^m} + \mathcal{O}((\Delta x)^2)$$

forward difference operator: $u \xrightarrow{\text{map}} \Delta_{1+}[u]$

$$\Delta_{1+}[u] = \mathcal{S}_{\Delta x} - \mathbb{I}, \quad \mathcal{S}_{\Delta x} \stackrel{d}{=} u(x + \Delta x)$$

$\xrightarrow{\text{shift operator}}$ \nwarrow identity operator

$$\Delta_{1+}^m = \Delta_{1+}(\Delta_{1+}^{m-1})$$

$$\Delta_{1+}^m = (\mathcal{S}_{\Delta x} - \mathbb{I})^m$$

linear: $\Delta_{1+}[\alpha u + \beta v](x) = \alpha \Delta_{1+}[u](x) + \beta \Delta_{1+}[v](x)$

Leibniz rule: $\Delta_{1+}[uv](x) = \Delta_{1+}[u] \cdot v(x + \Delta x) + u(x) \cdot \Delta_{1+}[v](x)$

et cetera ...

Finite differences [4]

Taylor series: $\Delta_{1+} = \Delta x \cdot D + \frac{1}{2!} (\Delta x)^2 D^2 + \frac{1}{3!} (\Delta x)^3 D^3 + \dots = e^{\Delta x D} - I$

↑
operator: $u \rightarrow u'$

(valid for sufficiently small Δx)

$\Rightarrow \int_{\Delta x}^{\prime} = e^{\Delta x D}$ and $\ln(1 + \Delta_{1+}) = \Delta_{1+} - \frac{1}{2} \Delta_{1+}^2 - \frac{1}{3} \Delta_{1+}^3 + \dots$

in a similar way: $\Delta x D = -\ln(1 - \Delta_{-})$ and $\Delta x D = 2 \operatorname{arcsinh}(\frac{1}{2} \Delta_{1c})$

rules: $\Delta_{1+} c = 0$, $\Delta_{1+} (\alpha u + \beta v) = \alpha \Delta_{1+} u + \beta \Delta_{1+} v$

$$\Delta_{1+} (uv) = u \Delta_{1+} v + v \Delta_{1+} u + \Delta_{1+} u \Delta_{1+} v$$

$$\Delta_{1-} (uv) = u \Delta_{1-} v + v \Delta_{1-} u - \Delta_{1-} u \Delta_{1-} v$$

$$\Delta_{1-} \left(\frac{u}{v} \right) = \frac{v \Delta_{1-} u - u \Delta_{1-} v}{v \cdot (v - \Delta_{1-} v)}$$

etcetera -----

Finite differences [5]

Method of undetermined coefficients:

Example 1: $u'(x_i) \approx Au_i + Bu_{i-1} + Cu_{i-2}$ (*)

$$\stackrel{\text{d}}{=} u_{x,i}$$

one-sided approximation
on $\{x_{i-2}, x_{i-1}, x_i\}$
"stencil"

A, B, C to be determined

Taylor expansions:

$$\begin{cases} u_{i-1} = u_i + (-\Delta x)u_x + \frac{(-\Delta x)^2}{2}u_{xx} + \frac{(-\Delta x)^3}{6}u_{xxx} + \dots \\ u_i = u_i + 0 \cdot \Delta x + 0 \cdot (\Delta x)^2 + \dots \\ u_{i-2} = u_i + (-2\Delta x)u_x + \frac{(-2\Delta x)^2}{2}u_{xx} + \frac{(-2\Delta x)^3}{6}u_{xxx} + \dots \end{cases}$$

substitute
in (*)

$$\Rightarrow Au_i + Bu_{i-1} + Cu_{i-2} = Au_i + Bu_i - B\Delta x u_x + \frac{B}{2}(\Delta x)^2 u_{xx} - \frac{B}{6}(\Delta x)^3 u_{xxx} + \dots + Cu_i - 2C\Delta x u_x + 2C(\Delta x)^2 u_{xx} - \frac{4}{3}(\Delta x)^3 u_{xxx} + \dots$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ -B\Delta x - 2C\Delta x = 1 \\ \frac{B}{2}(\Delta x)^2 + 2C(\Delta x)^2 = 0 \end{cases} \xrightarrow{\text{solve}} \begin{cases} A = \frac{3}{2\Delta x} \\ B = -\frac{2}{\Delta x} \\ C = \frac{1}{2\Delta x} \end{cases}$$

Finite differences [6]

$$\Rightarrow u_{x,i} = \frac{1}{2\Delta x} [3u_i - 4u_{i-1} + u_{i-2}]$$

the error: $\underbrace{u_{x,i}}_{\text{approx.}} - \underbrace{u'(x_i)}_{\text{exact}} = -B \frac{(\Delta x)^3}{6} u_{xxx} + C \frac{(-2\Delta x)^3}{6} u_{xxx} + \text{H.O.T.}$

(higher order terms)

$$= -\frac{1}{3} (\Delta x)^2 u_{xxx} + \mathcal{O}(\Delta x)^3$$

↑
second-order

" "
 $\mathcal{O}(\Delta x)^4$

Example 2: [a 4th order approximation for 1st derivative
stencil: $\{x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$]

Taylor:

$$u_{i-2} = u_i + u_x (-2\Delta x) + \frac{u_{xx}}{2} (-2\Delta x)^2 + \dots$$

$$u_{i-1} = u_i + u_x (-\Delta x) + \frac{u_{xx}}{2} (-\Delta x)^2 + \dots$$

$$u_i = u_i + 0 \cdot \Delta x + 0 \cdot (\Delta x)^2 + \dots$$

$$u_{i+1} = u_i + u_x (\Delta x) + \frac{u_{xx}}{2} (\Delta x)^2 + \dots$$

$$u_{i+2} = u_i + u_x (2\Delta x) + \frac{u_{xx}}{2} (2\Delta x)^2 + \dots$$

Finite differences [7]

$$\text{Set: } u_{x,i} = A u_{i-2} + B u_{i-1} + \overset{\substack{\uparrow \\ \text{suppose}}}{0} \cdot u_i + C \cdot u_{i+1} + D \cdot u_{i+2}$$

$$\Rightarrow \begin{cases} -2A - B + C + 2D = 1/\Delta x \\ 4A + B + C + 4D = 0 \\ -8A - B + C + 8D = 0 \\ 16A + B + C + 16D = 0 \end{cases} \begin{array}{l} \text{(eliminate 2}^{\text{nd}} \text{ order derivative terms)} \\ \text{(" 3}^{\text{rd}} \text{ " " " ")} \\ \text{(" 4}^{\text{th}} \text{ " " " ")} \end{array}$$

$$\Rightarrow A = \frac{2}{4!\Delta x}, B = \frac{-16}{4!\Delta x}, C = \frac{16}{4!\Delta x}, D = \frac{-2}{4!\Delta x} \quad \text{error: } O((\Delta x)^4)$$

In general: $\begin{cases} \text{given (small) } \Delta x, m^{\text{th}} \text{ order derivative} \\ \text{error} = O((\Delta x)^p) \end{cases}$

$$\frac{(\Delta x)^m}{m!} u_i^{(m)} = \sum_{i=i_{\min}}^{i=i_{\max}} C_i u_i$$

$\vec{C} = (C_{i_{\min}}, \dots, C_{i_{\max}})$ is called the template
or convolution mask
for the approximation

Finite differences [8]

select m and p and find C_i : $u_{i+1} = \sum_{n=0}^{\infty} \frac{i^n}{n!} (\Delta x)^n u_i^{(n)}$ (Taylor)

derivative to be approximated \swarrow order of approximation \nearrow

$$\Rightarrow u(x_i) = \frac{m!}{(\Delta x)^m} \sum_{n=0}^{m+p-1} \left[\sum_{i=i_{\min}}^{i_{\max}} i^n C_i \right] \frac{(\Delta x)^n}{n!} u_i^{(n)} + \underbrace{O((\Delta x)^p)}_{\text{error}}$$

approximation

$$\Rightarrow \sum_{i=i_{\min}}^{i_{\max}} i^n C_i = \begin{cases} 0 & , 0 \leq n \leq m+p-1, n \neq m \\ 1 & , n=m \end{cases}$$

\Rightarrow a set of $m+p$ linear equations in $i_{\max} - i_{\min} + 1$ unknowns

* if we restrict the # unknowns to $m+p$, then linear system has a unique solution

Examples: $i_{\min}=0, i_{\max}=m+p-1$ (forward FD's)
 $i_{\max}=0, i_{\min}=-(m+p-1)$ (backward FD's)
 $i_{\max} = -i_{\min} \approx (m+p-1)/2$ (central FD's)

Finite differences [9]

Table

m	p	type	i_{\min}	i_{\max}	formula	error
1	1	forward	0	1	$(u_{i+1} - u_i) / \Delta x$	$O(\Delta x)$
1	1	backward	-1	0	$(u_i - u_{i-1}) / \Delta x$	$O(\Delta x)$
1	2	central	-1	1	$(u_{i+1} - u_{i-1}) / (2\Delta x)$	$O((\Delta x)^2)$
1	2	forward	0	2	$(-u_{i+2} + 4u_{i+1} - 3u_i) / (2\Delta x)$	$O((\Delta x)^2)$
1	2	backward	-2	0	$(3u_i - 4u_{i-1} + u_{i-2}) / (2\Delta x)$	$O((\Delta x)^2)$
1	4	central	-2	2	$(-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}) / (12\Delta x)$	$O((\Delta x)^4)$
2	1	forward	0	2	$(u_{i+2} - 2u_{i+1} + u_i) / (\Delta x)^2$	$O(\Delta x)$
2	2	central	-1	1	$(u_{i+1} - 2u_i + u_{i-1}) / (\Delta x)^2$	$O((\Delta x)^2)$
2	4	central	-2	2	$(-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}) / (12\Delta x)^2$	$O((\Delta x)^4)$

Finite difference matrices [1]

approximate $\frac{\partial}{\partial x}$ on the stencil $\{x_{i-1}, x_i, x_{i+1}\}$, uniform Δx

define the matrix $D_{ic} = \frac{1}{2\Delta x}$

and $\vec{u} = (\dots, u_{i-1}, u_i, u_{i+1}, \dots)$

this corresponds to

$$u_x(x_i) \approx u_{x,i} = \frac{u_{i+1} - u_{i-1}}{2\Delta x} \quad (O((\Delta x)^2) \text{ error})$$

$$i=2: u_{x,2} = \frac{u_2 - u_1}{2\Delta x} \leftarrow \mathcal{F}$$

$$i=1: u_{x,1} = \frac{u_2 - u_0}{2\Delta x} \leftarrow ?$$

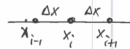
~~$$i=0: u_{x,0} = \frac{u_1 - u_{-1}}{2\Delta x} \leftarrow ??$$~~

$$\text{similarly: } i=M-1: u_{x,M-1} = \frac{u_M - u_{M-2}}{2\Delta x} \leftarrow \mathcal{F}$$

$$i=M: u_{x,M} = \frac{u_{M+1} - u_{M-1}}{2\Delta x} \leftarrow ?$$

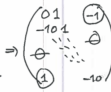


boundary conditions of the DE



periodic BC: $u_0 = u_{M+1}$

$$\Rightarrow \begin{cases} u_{x,1} = \frac{u_2 - u_{M+1}}{2\Delta x} \\ u_{x,M} = \frac{u_0 - u_{M-1}}{2\Delta x} \end{cases}$$



skew-symmetric (a circulant matrix)
eigenvalues on imaginary axis

Finite difference matrices [2]

forward FD: $u_{x,i} = \frac{u_{i+1} - u_i}{\Delta x}$ ($\mathcal{O}(\Delta x)$ error) $\Rightarrow \mathcal{D}_{1+} = \frac{1}{\Delta x} \begin{pmatrix} -1 & 1 & 0 & \dots & \textcircled{0} \\ 0 & -1 & 1 & \dots & \\ \ominus & & -1 & 1 & \ominus \\ & & & \ddots & \\ & & & & -1 & 1 \end{pmatrix}$

backward FD: $u_{x,i} = \frac{u_i - u_{i-1}}{\Delta x}$ ($\mathcal{O}(\Delta x)$ error) $\Rightarrow \mathcal{D}_{1-} = \frac{1}{\Delta x} \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & & -1 & 1 & \\ \ominus & & & \ddots & \\ & & & & -1 & 1 \end{pmatrix}$

FD matrix for $\frac{\partial^2 u}{\partial x^2}$ at $x = x_i$ with periodic BCs

$$\mathcal{D}_{2c} = \mathcal{D}_{1+} \mathcal{D}_{1-} = \mathcal{D}_{1-} \mathcal{D}_{1+} = \frac{1}{(\Delta x)^2} \begin{pmatrix} -2 & 1 & & & \textcircled{1} \\ 1 & -2 & 1 & & \\ \ominus & & -2 & 1 & \ominus \\ & & & \ddots & \\ & & & & -2 & 1 \end{pmatrix} \quad \begin{array}{l} \text{(a symmetric matrix)} \\ \downarrow \\ \text{real eigenvalues} \end{array}$$

\mathcal{D}_{2c} with homogeneous Dirichlet BCs: $u_0 = u_N = 0$

$$u_{xx,i} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} \quad (\mathcal{O}(\Delta x^2) \text{ error})$$

$i=1: (u_2 - 2u_1 + u_0) / (\Delta x)^2$

$i=N-1: (u_N - 2u_{N-1} + u_{N-2}) / (\Delta x)^2$

$$\Rightarrow \frac{1}{(\Delta x)^2} \begin{pmatrix} -2 & & & & \\ 1 & -2 & 1 & & \\ & & -2 & 1 & \\ & & & \ddots & \\ & & & & -2 & 1 \end{pmatrix}$$

Eigenvalues of FD matrices [1]

tridiagonal matrix A (size: $M-1 \times M-1$)

$$A = \begin{pmatrix} a & b & & & \\ b & a & b & & \\ & b & a & b & \\ \emptyset & & \ddots & & \\ & & & b & a & b \\ & & & & b & a \end{pmatrix}$$

$$\Rightarrow \lambda_j(A) = a + 2b \cos\left(\frac{j\pi}{M}\right)$$

$$j = 1, 2, \dots, M-1$$

$$\& \text{eigenvectors: } \vec{v}_j(A) = \begin{pmatrix} v_j^1 \\ v_j^2 \\ \vdots \\ v_j^{M-1} \end{pmatrix}$$

$$\text{with } v_j^k = \sin\left(\frac{kj\pi}{M}\right), \quad k, j \in \{1, \dots, M-1\}$$

special case: $a = -\frac{2}{(\Delta x)^2}$, $b = \frac{1}{(\Delta x)^2}$ ($A = D_{2c}$)

$$\Rightarrow \lambda_j(D_{2c}) = \frac{2}{(\Delta x)^2} \left(\cos\left(\frac{j\pi}{M}\right) - 1 \right), \quad j = 1, \dots, M-1$$

how to find the λ 's?

define $\tilde{A} = \begin{pmatrix} -2 & 1 & & \emptyset \\ 1 & -2 & 1 & \\ \emptyset & & \ddots & \\ \emptyset & & & 1 & -2 \end{pmatrix}$ (the factor $\frac{1}{(\Delta x)^2}$ can be added at the end of the calculation)

$$\tilde{A} \vec{y} = \lambda \vec{y}$$

Eigenvalues of FD matrices [2]

$$\Leftrightarrow \begin{cases} y_{j-1} - (z + \lambda)y_j + y_{j+1} = 0, & j = 1, \dots, M-1 \\ y_0 = y_M = 0 \end{cases}$$

general solution of this 2nd order recurrence relation can be written as:

$$y_j = \alpha w_1^j + \beta w_2^j \quad \text{where } w_1, w_2 \text{ are solutions of:}$$

(substitute $y_j = z^j$) $z^2 - (z + \lambda)z + 1 = 0$

from $y_0 = 0$ follows: $\alpha + \beta = 0$

from $y_M = 0$ follows: $\alpha w_1^M + \beta w_2^M = 0$

$$\Leftrightarrow \begin{cases} \beta = -\alpha \\ w_1^M = w_2^M \end{cases} \Leftrightarrow \begin{cases} \beta = -\alpha \\ \left(\frac{w_1}{w_2}\right)^M = 1 \end{cases}$$

$$\Rightarrow y_j = \alpha (w_1^j - w_2^j) \quad \text{with} \quad \frac{w_1}{w_2} = e^{\frac{2\pi i k}{M}}, \quad k = 1, 2, \dots, M-1$$

we may assume $w_1 = e^{\frac{\pi i k}{M}}$ and $w_2 = e^{-\frac{\pi i k}{M}}$ for some value of k (note $w_1 \cdot w_2 = 1$)

the number λ is an eigenvalue of \tilde{A} , if w_1, w_2 satisfy: $z^2 - (z + \lambda)z + 1 = 0$

Since $(z - w_1)(z - w_2) = z^2 - (w_1 + w_2)z + w_1 w_2 = 0$,

$$\text{we find: } \begin{cases} w_1 + w_2 = z + \lambda \\ w_1 w_2 = 1 \end{cases} \Rightarrow \lambda = w_1 + w_2 - z = e^{\frac{\pi i k}{M}} + e^{-\frac{\pi i k}{M}} - z$$

Eigenvalues of FD matrices [3]

$$= 2 \cos\left(\frac{\pi k}{M}\right) - 2$$

$$= -4 \sin^2\left(\frac{\pi k}{2M}\right)$$

$$= -4 \sin^2\left(\frac{\pi k \Delta x}{2}\right)$$

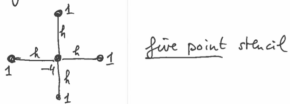
$$\Rightarrow \lambda(A) = -\frac{4}{(\Delta x)^2} \sin^2\left(\frac{\pi k \Delta x}{2}\right)$$

the eigenvectors of A follow from $y_j = \alpha (w_1^j - w_2^j)$
= ...

(see also the document on the webpage)

Higher dimensions [1]

2D stencil for the Laplacian Δ with $\Delta x = \Delta y = h$:



we need 2D Taylor expansions and two indices i, j :

$$\left\{ \begin{array}{l} u_{i+1,j} = u_{i,j} + h u_x + \frac{h^2}{2} u_{xx} + \dots \\ u_{i,j-1} = u_{i,j} - h u_y + \frac{h^2}{2} u_{yy} + \dots \\ \dots \end{array} \right. \Rightarrow \Delta u_{i,j} \approx \frac{1}{h^2} \left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4 u_{i,j} \right]$$

error: $\mathcal{O}(h^2)$

a nine point stencil



Note: $u_{i+1,j+1} = u_{i,j} + h u_x + h u_y + \frac{h^2}{2} (u_{xx} + 2u_{xy} + u_{yy}) + \dots$

$$\Rightarrow \Delta u_{i,j} \approx \frac{1}{6h^2} \left[4u_{i+1,j} + 4u_{i-1,j} + 4u_{i,j+1} + 4u_{i,j-1} + u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1} - 20 u_{i,j} \right]$$

error: $\mathcal{O}(h^2)$? \rightarrow exercise

Higher dimensions [2]

3D: seven point stencil $\Delta x = \Delta y = \Delta z = h$



$$\Delta u_{i,j,k} \approx \frac{1}{h^2} [u_{i+1,j,k} + \dots + u_{i,j,k-1} - 6u_{i,j,k}] \quad \text{error: } O(h^2)$$

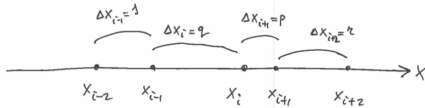
$$\begin{array}{c} \vdots \\ \downarrow \\ \text{dD} \end{array} \Delta u_{i_1, i_2, \dots, i_d} \approx \frac{1}{h^2} \left[\underbrace{u_{i_1+1, i_2, \dots, i_d} + \dots + u_{i_1, i_2, \dots, i_d-1}}_{\text{2d terms}} - 2d \cdot u_{i_1, i_2, \dots, i_d} \right]$$

mixed derivative:
(u 2D)



$$\frac{\partial^2 u}{\partial x \partial y}_{i,j} \approx \frac{u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1}}{4h^2}$$

Non-uniform grids [1]



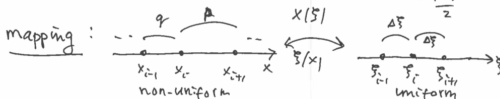
if $p = q = r = s = \dots = \Delta x$, the uniform grid

example: $\frac{\partial u}{\partial x}$ at $x = x_i$ (central FD, stencil: $\{x_{i-1}, x_i, x_{i+1}\}$)

$$\begin{aligned} \text{Taylor: } u_{i+1} &= u_i + p u_x + \frac{p^2}{2} u_{xx} + \dots \\ u_i &= u_i + 0 + 0 + \dots \\ u_{i-1} &= u_i - q u_x + \frac{q^2}{2} u_{xx} - \dots \end{aligned}$$

$$\Rightarrow u_{x,i} = \frac{u_{i+1} - u_{i-1}}{p+q} \quad \text{with error: } - \frac{p^2 - q^2}{2(p+q)} u_{xx} - \frac{p^3 + q^3}{6(p+q)} u_{xxx} - \dots$$

$$= \frac{p-q}{2}$$



Non-uniform grids [2]

define $\Delta \xi = H = \frac{1}{M}$ ← number of gridpoints

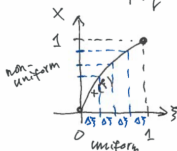
$$\begin{aligned}
 p = \Delta x_{i+1} = x_{i+1} - x_i &= x(\xi_{i+1}) - x(\xi_i) \\
 &= \Delta \xi \cdot x_\xi + \frac{(\Delta \xi)^2}{2} x_{\xi\xi} + \mathcal{O}((\Delta \xi)^3) \\
 &= H \cdot x_\xi + \frac{1}{2} H^2 x_{\xi\xi} + \mathcal{O}(H^3)
 \end{aligned}$$

$$\begin{aligned}
 q = \Delta x_{i-1} = x_i - x_{i-1} &= x(\xi_i) - x(\xi_{i-1}) \\
 &= H \cdot x_\xi - \frac{1}{2} H^2 x_{\xi\xi} + \mathcal{O}(H^3)
 \end{aligned}$$

$$\Rightarrow p - q = H^2 x_{\xi\xi} + \mathcal{O}(H^4) \quad \text{! (check!)}$$

$$p + q = H^2 x_\xi^2 + \mathcal{O}(H^4)$$

$$\Rightarrow u_{x_i} = \frac{u_{i+1} - u_{i-1}}{p + q} \quad \text{with error: } -\frac{H^2}{6} [3x_{\xi\xi} u_{xx} + x_\xi^2 u_{xxx}] + \mathcal{O}(H^4)$$



$x(\xi) = \xi$: uniform grid in x

$x_\xi > 0$: mapping to non-singular

x_ξ and $x_{\xi\xi}$ play an important role!

second order

properties of mapping (↔ properties of non-uniform grid)

how to choose $x(\xi)$?

→ lecture on adaptive grids

Outlook to Lecture 3

- ⤴ prepare exercises of Lecture 2 (see webpage!)
- † Method-of-Lines (horizontal vs vertical)
- Υ time-integration methods
- ⊕ stability regions