Lecture 6

Paul Andries Zegeling

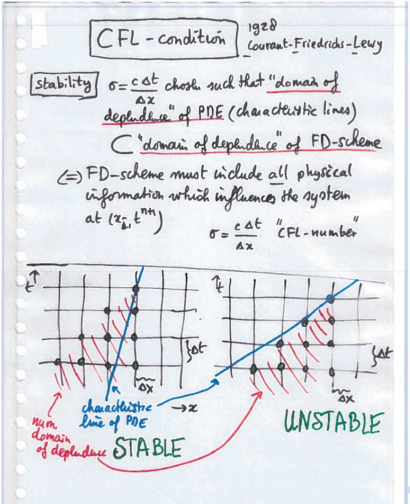
Department of Mathematics, Utrecht University

Numerical Methods for Time-Dependent PDEs

Outline of Lecture 6

- exercises of Lecture 5
- nonlinear hyperbolic PDEs
- CFL-condition
- wave equation
- CTCS
- extra IC & CFL-condition
- outlook to Lecture 7

CFL condition [1]



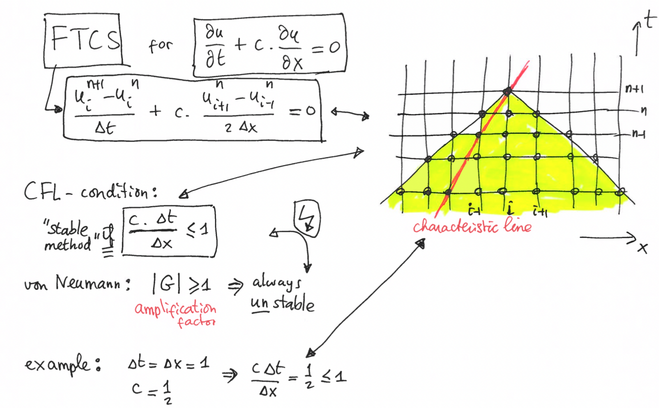
CFL - condition 1928 Courant-Friedrichs-Lewy

stability $\sigma = \frac{c\Delta t}{\Delta x}$ chosen such that "domain of dependence" of PDE (characteristic lines) $\subset$ "domain of dependence" of FD-scheme

$\Leftrightarrow$ FD-scheme must include all physical information which influences the system at (x_i, t^{n+1})

$\sigma = \frac{c\Delta t}{\Delta x}$ "CFL-number"

CFL condition [2]



$\boxed{\text{FTCS}}$ for $\boxed{\frac{\partial u}{\partial t} + c\cdot\frac{\partial u}{\partial x} = 0}$

$$\boxed{\frac{u_i^{n+1}-u_i^{n}}{\Delta t} + c\cdot\frac{u_{i+1}^{n}-u_{i-1}^{n}}{2\,\Delta x} = 0}$$

CFL-condition:

"stable method" $\underline{\underline{\text{if}}}$ $\boxed{\frac{c\cdot\Delta t}{\Delta x} \le 1}$

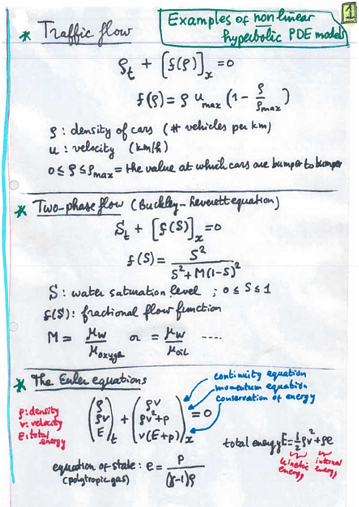
↯

von Neumann: $|G| \ge 1 \Rightarrow$ always $\underline{\text{un}}$stable

amplification factor

example: $\Delta t = \Delta x = 1$, $c = \frac{1}{2}$ $\Rightarrow \frac{c\,\Delta t}{\Delta x} = \frac{1}{2} \le 1$

Nonlinear hyperbolic PDEs [1]



Examples of non linear hyperbolic PDE models

1

* Traffic flow

$$\rho_t + [f(\rho)]_x = 0$$

$$f(\rho) = \rho\, u_{max} \left(1 - \frac{\rho}{\rho_{max}}\right)$$

ρ : density of cars (# vehicles per km)
u : velocity (km/h)
$0 \leq \rho \leq \rho_{max}$ = the value at which cars are bumper to bumper

* Two-phase flow (Buckley-Leverett equation)

$$S_t + [f(S)]_x = 0$$

$$f(S) = \frac{S^2}{S^2 + M(1-S)^2}$$

S : water saturation level ; $0 \leq S \leq 1$
$f(S)$: fractional flow function

$$M = \frac{\mu_w}{\mu_{oxygen}} \quad \text{or} \quad = \frac{\mu_w}{\mu_{oil}} \quad \ldots$$

* The Euler equations

ρ: density
v: velocity
E: total energy

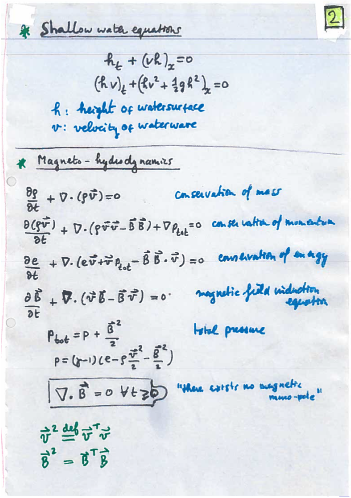
$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_x = 0$$

continuity equation
momentum equation
conservation of energy

total energy $E = \underbrace{\frac{1}{2}\rho v^2}_{\text{kinetic energy}} + \underbrace{\rho e}_{\text{internal energy}}$

equation of state : $e = \dfrac{p}{(\gamma-1)\rho}$
(polytropic gas)

Nonlinear hyperbolic PDEs [2]



$\boxed{2}$

* Shallow water equations

$$h_t + (vh)_x = 0$$

$$(hv)_t + \left(hv^2 + \tfrac{1}{2}gh^2\right)_x = 0$$

h : height of watersurface

v : velocity of waterwave

* Magneto-hydrodynamics

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$ conservation of mass

$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v} - \vec{B}\vec{B}) + \nabla p_{tot} = 0$$ conservation of momentum

$$\frac{\partial e}{\partial t} + \nabla\cdot(e\vec{v} + \vec{v}p_{tot} - \vec{B}\vec{B}\cdot\vec{v}) = 0$$ conservation of energy

$$\frac{\partial \vec{B}}{\partial t} + \nabla\cdot(\vec{v}\vec{B} - \vec{B}\vec{v}) = 0$$ magnetic field induction equation

$$p_{tot} = p + \frac{\vec{B}^2}{2}$$ total pressure

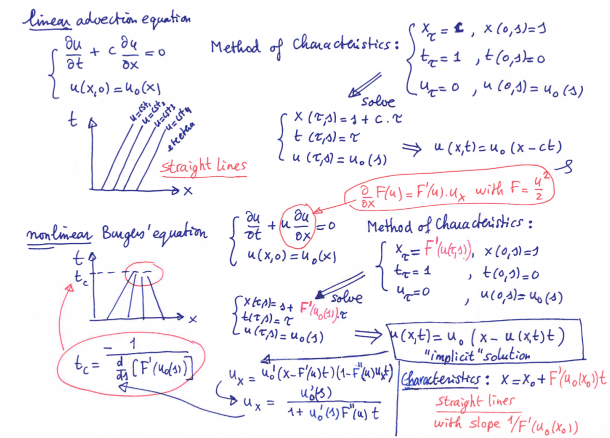
$$p = (\gamma-1)\left(e - \rho\frac{\vec{v}^2}{2} - \frac{\vec{B}^2}{2}\right)$$

$$\boxed{\nabla\cdot\vec{B} = 0 \;\; \forall t \geq 0}$$ "there exists no magnetic mono-pole"

$$\vec{v}^2 \overset{def}{=} \vec{v}^T\vec{v}$$

$$\vec{B}^2 = \vec{B}^T\vec{B}$$

Nonlinear hyperbolic PDEs [3]



linear advection equation

$$\begin{cases} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Method of Characteristics: $\begin{cases} x_\tau = c, & x(0,s) = s \\ t_\tau = 1, & t(0,s) = 0 \\ u_\tau = 0, & u(0,s) = u_0(s) \end{cases}$

solve

$$\begin{cases} x(\tau,s) = s + c.\tau \\ t(\tau,s) = \tau \\ u(\tau,s) = u_0(s) \end{cases} \Rightarrow u(x,t) = u_0(x-ct)$$

$\frac{\partial}{\partial x}F(u) = F'(u).u_x$ with $F = \frac{u^2}{2}$

nonlinear Burgers' equation

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Method of Characteristics:

$$\begin{cases} x_\tau = F'(u(\tau,s)), & x(0,s) = s \\ t_\tau = 1, & t(0,s) = 0 \\ u_\tau = 0, & u(0,s) = u_0(s) \end{cases}$$

solve

$$\begin{cases} x(\tau,s) = s + F'(u_0(s)).\tau \\ t(\tau,s) = \tau \\ u(\tau,s) = u_0(s) \end{cases} \Longrightarrow \boxed{u(x,t) = u_0(x - u(x,t)t)}$$

"implicit" solution

$$t_c = \frac{-1}{\frac{d}{ds}[F'(u_0(s))]}$$

$$u_x = u_0'(x - F'(u)t)(1 - F''(u)u_x t)$$

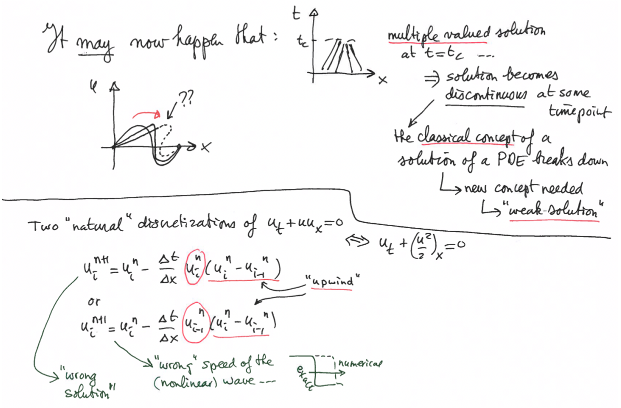
$$u_x = \frac{u_0'(s)}{1 + u_0'(s)F''(u)\,t}$$

Characteristics: $x = x_0 + F'(u_0(x_0))t$

straight lines

with slope $1/F'(u_0(x_0))$

Nonlinear hyperbolic PDEs [4]



It may now happen that:

multiple valued solution at $t=t_c$ ...

$\Rightarrow$ solution becomes discontinuous at some timepoint

the classical concept of a solution of a PDE breaks down

$\hookrightarrow$ new concept needed

$\hookrightarrow$ "weak-solution"

Two "natural" discretizations of $u_t+uu_x=0$ $\Leftrightarrow$ $u_t+\left(\frac{u^2}{2}\right)_x=0$

$$u_i^{n+1}=u_i^n-\frac{\Delta t}{\Delta x}\boxed{u_i^n}\underline{(u_i^n-u_{i-1}^n)}$$

"upwind"

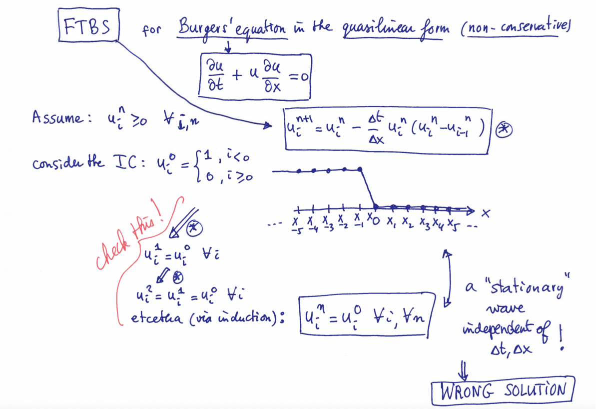
or

$$u_i^{n+1}=u_i^n-\frac{\Delta t}{\Delta x}\boxed{u_{i-1}^n}\underline{(u_i^n-u_{i-1}^n)}$$

"wrong solution"

"wrong" speed of the (nonlinear) wave ---

Nonlinear hyperbolic PDEs [5]



$\boxed{\text{FTBS}}$ for Burgers' equation in the quasilinear form (non-conservative)

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$

Assume: $u_i^n \geq 0 \quad \forall i,n$

$$\boxed{u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n \left(u_i^n - u_{i-1}^n\right)} \quad (*)$$

consider the IC: $u_i^0 = \begin{cases} 1, & i<0 \\ 0, & i \geq 0 \end{cases}$

check this!

$(*)$

$u_i^1 = u_i^0 \quad \forall i$

$(*)$

$u_i^2 = u_i^1 = u_i^0 \quad \forall i$

etcetera (via induction): $\boxed{u_i^n = u_i^0 \quad \forall i, \forall n}$

a "stationary" wave independent of $\Delta t, \Delta x$!

$\Downarrow$

$\boxed{\text{WRONG SOLUTION}}$

Conservative form [1]

$$\boxed{u_t + \left(F(u)\right)_x = 0}$$

example: $\boxed{F(u) = \frac{u^2}{2}}$

$\left(\frac{u^2}{2}\right)_x$ conservative form

uu_x non-conservative form

1) $u_i^{n+1} = u_i^n - \Delta t \cdot \dfrac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{4\,\Delta x}$

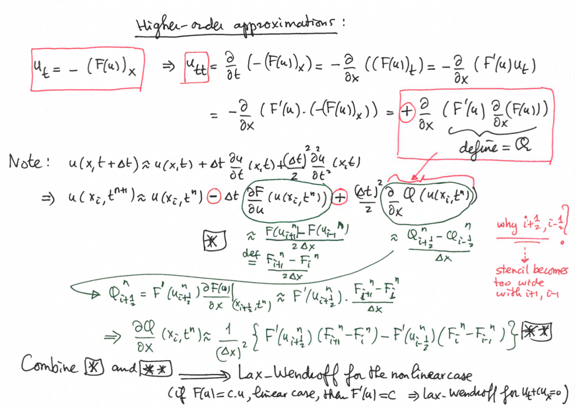
2) $u_i^{n+1} = u_i^n - \Delta t\, u_i^n \dfrac{(u_{i+1}^n - u_{i-1}^n)}{2\,\Delta x}$

3) $u_i^{n+1} = u_i^n - \Delta t\; u_i^n \dfrac{(u_i^n - u_{i-1}^n)}{\Delta x}$

4) $u_i^{n+1} = \dfrac{u_{i+1}^n + u_{i-1}^n}{2} - \dfrac{\Delta t}{4\,\Delta x}\left((u_{i+1}^n)^2 - (u_{i-1})^n\right)^2$ "Lax-Friedrichs"

etcetera (it matters!)

Conservative form [2]



Higher-order approximations:

$$\boxed{u_t = -(F(u))_x} \Rightarrow \boxed{u_{tt}} = \frac{\partial}{\partial t}\left(-(F(u))_x\right) = -\frac{\partial}{\partial x}\left((F(u))_t\right) = -\frac{\partial}{\partial x}\left(F'(u)u_t\right)$$

$$= -\frac{\partial}{\partial x}\left(F'(u)\cdot(-(F(u))_x)\right) = \boxed{\textcircled{+}\frac{\partial}{\partial x}\Big(\underbrace{F'(u)\frac{\partial}{\partial x}(F(u))}_{\text{define } = \mathcal{Q}}\Big)}$$

Note: $u(x,t+\Delta t) \approx u(x,t) + \Delta t\frac{\partial u}{\partial t}(x,t) + \frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}(x,t)$

$$\Rightarrow u(x_i,t^{n+1}) \approx u(x_i,t^n) \ominus \Delta t\frac{\partial F}{\partial u}(u(x_i,t^n)) \oplus \frac{(\Delta t)^2}{2}\frac{\partial}{\partial x}\mathcal{Q}(u(x_i,t^n))$$

$$\boxed{*} \quad \approx \frac{F(u_{i+1}^n) - F(u_{i-1}^n)}{2\Delta x} \overset{\text{def}}{=} \frac{F_{i+1}^n - F_i^n}{2\Delta x} \qquad \approx \frac{\mathcal{Q}_{i+\frac{1}{2}}^n - \mathcal{Q}_{i-\frac{1}{2}}^n}{\Delta x}$$

why $i+\frac{1}{2}, i-\frac{1}{2}$?

stencil becomes too wide with $i+1$, $i-1$

$$\mathcal{Q}_{i+\frac{1}{2}}^n = F'(u_{i+\frac{1}{2}}^n)\frac{\partial F(u)}{\partial x}\Big|_{(x_{i+\frac{1}{2}},t^n)} \approx F'(u_{i+\frac{1}{2}}^n)\cdot\frac{F_{i+1}^n - F_i^n}{\Delta x}$$

$$\Rightarrow \frac{\partial \mathcal{Q}}{\partial x}(x_i,t^n) \approx \frac{1}{(\Delta x)^2}\left\{F'(u_{i+\frac{1}{2}}^n)(F_{i+1}^n - F_i^n) - F'(u_{i-\frac{1}{2}}^n)(F_i^n - F_{i-1}^n)\right\} \boxed{**}$$

Combine $\boxed{*}$ and $\boxed{**}$ $\Longrightarrow$ Lax–Wendroff for the nonlinear case

(if $F(u) = c.u$, linear case, then $F'(u) = c$ $\Rightarrow$ lax-wendroff for $u_t + cu_x = 0$)

Finite volumes [1]

Finite Volumes

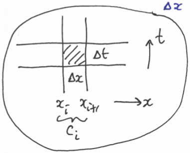
Rather than viewing u_i^n as an approximation to the single value $u(x_i,t^n)$, we will now view it as an approximation to the average value of u over an interval $C_i = [x_i, x_{i+1}]$

"cell", "volume"

with $x_i = x_L + \frac{i-1}{N}$, $i=1,\dots,N$

$\Delta x = x_{i+1} - x_i$

$$u_i^n \approx \frac{\int_{x_i}^{x_{i+1}} u(x,t^n)\,dx}{\Delta x} = \frac{1}{\Delta x}\int_{C_i} u(x,t^n)\,dx$$



$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(F(u)) = 0}$$

$$\downarrow$$

$$\iint_{\text{▨}} (u_t + (F(u))_x)\,dx\,dt = \underbrace{\iint_{\text{▨}} 0\,dx\,dt}_{=0}$$

$$\Leftrightarrow \int_{x_i}^{x_{i+1}}\int_{t^n}^{t^{n+1}} u_t\,dt\,dx + \int_{t^n}^{t^{n+1}}\int_{x_i}^{x_{i+1}} (F(u))_x\,dx\,dt = 0$$

Finite volumes [2]

$$\Leftrightarrow \int_{x_i}^{x_{i+1}} \left[u(x,t^{n+1}) - u(x,t^n) \right] dx + \int_{t^n}^{t^{n+1}} \left[F(u(x_{i+1},t)) - F(u(x_i,t)) \right] dt = 0$$

$$\Leftrightarrow \int_{C_i} u(x,t^{n+1})\,dx - \int_{C_i} u(x,t^n)\,dx = \int_{t^n}^{t^{n+1}} F(u(x_i,t))\,dt - \int_{t^n}^{t^{n+1}} F(u(x_{i+1},t))\,dt$$

still exact!!!

Re-arranging and dividing by Δx:

$$\underbrace{\frac{1}{\Delta x}\int_{C_i} u(x,t^{n+1})\,dx}_{\substack{u_i^{n+1} \approx \\ (\text{at time level } t^{n+1})}} = \underbrace{\frac{1}{\Delta x}\int_{C_i} u(x,t^{n})\,dx}_{\substack{u_i^{n} \approx \\ (\text{at time level } t^{n})}} - \frac{1}{\Delta x}\left\{ \int_{t^n}^{t^{n+1}} F(u(x_{i+1},t))\,dt - \int_{t^n}^{t^{n+1}} F(u(x_i,t))\,dt \right\}$$

$$\rightsquigarrow \underset{\text{"in conservative form"}}{\boxed{u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1}^n - F_i^n\right)}} \quad \text{with } F_i^n \underset{\uparrow\ \text{approximation}}{\approx} \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F(u(x_i,t))\,dt$$

Finite volumes [3]

Note (!)

"exact"

$$u_t + (F(u))_x = 0 \Leftrightarrow \int_0^t \int_{-\infty}^{\infty} u_t(x,s)\,dx\,ds + \int_0^t \int_{-\infty}^{\infty} (F(u))_x\,dx\,ds = \underbrace{\int_0^t \int_{-\infty}^{\infty} 0\,dx\,ds}_{=0}$$

$$\Leftrightarrow \int_{-\infty}^{\infty} \int_0^t u_t(x,s)\,ds\,dx + \int_0^t (F(u))\Big|_{x=-\infty}^{x=+\infty} ds = 0$$

$$\Leftrightarrow \int_{-\infty}^{\infty} [u(x,t) - u(x,0)]\,dx + \int_0^t \underbrace{[F(u(\infty,s)) - F(u(-\infty,s))]}_{\text{suppose} = 0}\,ds = 0$$

$\to u|_{\pm\infty} = 0$

or

$\to u|_{\infty} = u|_{-\infty}$

$\to$ ---

$\to$ example: $F(u) = u^2$

or F is an even function

$$\Leftrightarrow \int_{-\infty}^{\infty} [u(x,t) - u(x,0)]\,dx = 0$$

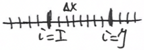
$$\Leftrightarrow \int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} \underbrace{u(x,0)}_{\text{initial condition}}\,dx$$

$\forall t \geq 0$

"conservation of mass"

Finite volumes [4]

Note ("numerical") sum $\Delta x\, u_i^{n+1}$ from $i=I$ to J over any set of grid cells:



$$\boxed{\Delta x \sum_{i=I}^{J} u_i^{n+1}} = \Delta x \sum_{i=I}^{J} u_i^n - \Delta t \sum_{i=I}^{J} \left(F_{i+1}^n - F_i^n\right)$$

$$= \Delta x \sum_{i=I}^{J} u_i^n - \Delta t \left\{ \cancel{F_{I+1}^n} - F_I^n + \cancel{F_{I+2}^n} - \cancel{F_{I+1}^n} + \cancel{\cdots} + \cancel{F_{J+1}^n} - F_J^n \right\}$$

$$= \boxed{\Delta x \sum_{i=I}^{J} u_i^n} - \Delta t \left[\underbrace{F_{J+1}^n - F_I^n}_{\text{"boundary" terms}} \right]$$

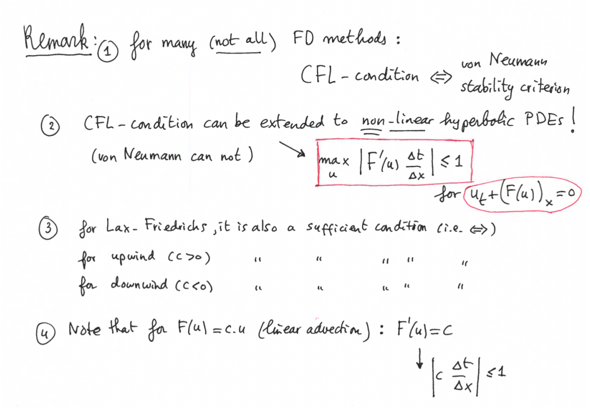
all terms disappear except

Conserved quantity $\sim \left[\int u\, dx\right]^{t^{n+1}}$ and $\sim \left[\int u\, dx\right]^{t^n}$

$I = 0$ (initial index) and $J = N$ (final index) with $F_{J+1} - F_I = F_{N+1} - F_0 = 0$

↑ as for continuous case

CFL vs von Neumann



Remark: ① for many (not all) FD methods :

CFL - condition ⇔ von Neumann stability criterion

② CFL-condition can be extended to non-linear hyperbolic PDEs!

(von Neumann can not) $\searrow$ $\boxed{\max_u \left| F'(u) \frac{\Delta t}{\Delta x} \right| \leq 1}$

for $u_t + (F(u))_x = 0$

③ for Lax-Friedrichs, it is also a sufficient condition (i.e. ⇔)

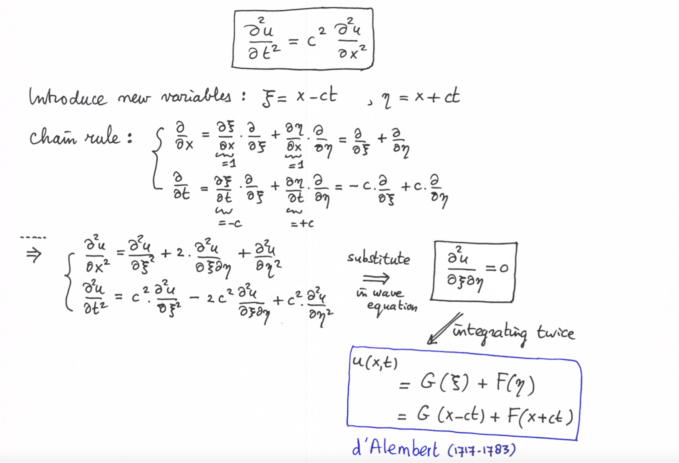
for upwind $(c>0)$ " " " " "

for downwind $(c<0)$ " " " " "

④ Note that for $F(u) = c.u$ (linear advection) : $F'(u) = c$

$$\downarrow \left| c \frac{\Delta t}{\Delta x} \right| \leq 1$$

Wave equation [1]



$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

Introduce new variables : $\xi = x - ct$, $\eta = x + ct$

Chain rule :

$$\begin{cases} \frac{\partial}{\partial x} = \underbrace{\frac{\partial \xi}{\partial x}}_{=1} \cdot \frac{\partial}{\partial \xi} + \underbrace{\frac{\partial \eta}{\partial x}}_{=1} \cdot \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial t} = \underbrace{\frac{\partial \xi}{\partial t}}_{=-c} \cdot \frac{\partial}{\partial \xi} + \underbrace{\frac{\partial \eta}{\partial t}}_{=+c} \cdot \frac{\partial}{\partial \eta} = -c \cdot \frac{\partial}{\partial \xi} + c \cdot \frac{\partial}{\partial \eta} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2 \cdot \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial \xi^2} - 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + c^2 \cdot \frac{\partial^2 u}{\partial \eta^2} \end{cases}$$

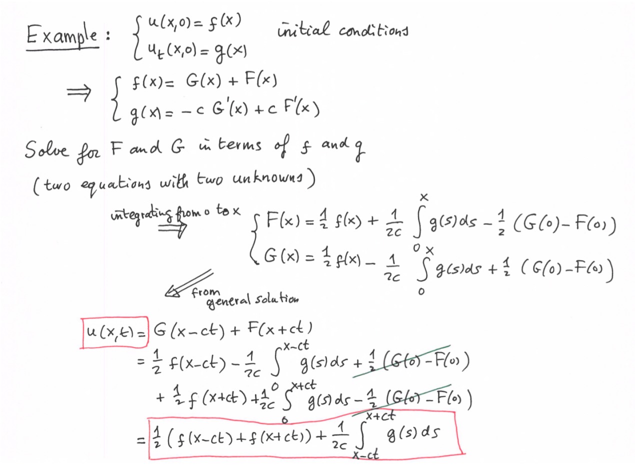
$$\xrightarrow[\text{in wave equation}]{\text{substitute}} \boxed{\frac{\partial^2 u}{\partial \xi \partial \eta} = 0}$$

integrating twice

$$\boxed{\begin{aligned} u(x,t) & \\ &= G(\xi) + F(\eta) \\ &= G(x-ct) + F(x+ct) \end{aligned}}$$

d'Alembert (1717-1783)

Wave equation [2]



Example: $\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$ initial conditions

$$\Rightarrow \begin{cases} f(x) = G(x) + F(x) \\ g(x) = -c\,G'(x) + c\,F'(x) \end{cases}$$

Solve for F and G in terms of f and g

(two equations with two unknowns)

$$\xrightarrow{\text{integrating from 0 to x}} \begin{cases} F(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(s)\,ds - \frac{1}{2}\left(G(0) - F(0)\right) \\ G(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(s)\,ds + \frac{1}{2}\left(G(0) - F(0)\right) \end{cases}$$

from general solution

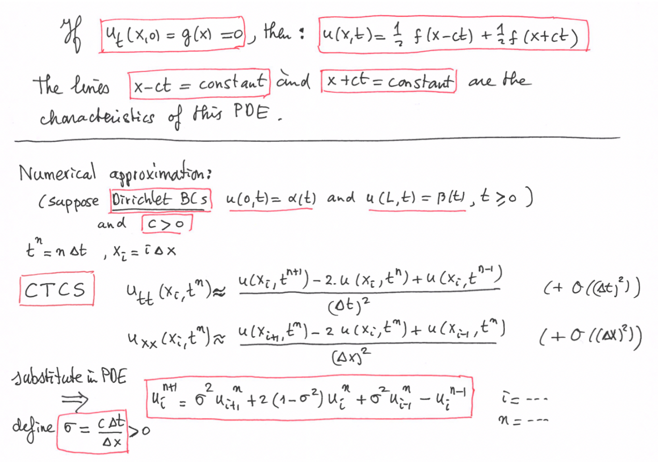
$$\boxed{u(x,t) =} G(x-ct) + F(x+ct)$$

$$= \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(s)\,ds + \cancel{\frac{1}{2}\left(G(0) - F(0)\right)}$$

$$+ \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(s)\,ds - \cancel{\frac{1}{2}\left(G(0) - F(0)\right)}$$

$$= \boxed{\frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\,ds}$$

Wave equation [3]



If $\boxed{u_t(x,0) = g(x) = 0}$, then: $\boxed{u(x,t) = \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct)}$

The lines $\boxed{x-ct = \text{constant}}$ and $\boxed{x+ct = \text{constant}}$ are the characteristics of this PDE.

Numerical approximation:

(suppose $\boxed{\text{Dirichlet BCs}}$ $\underline{u(0,t) = \alpha(t)}$ and $\underline{u(L,t) = \beta(t)}$, $t \geq 0$)

and $\boxed{c > 0}$

$t^n = n\Delta t$, $x_i = i\Delta x$

$\boxed{\text{CTCS}}$

$$u_{tt}(x_i,t^n) \approx \frac{u(x_i,t^{n+1}) - 2u(x_i,t^n) + u(x_i,t^{n-1})}{(\Delta t)^2} \quad (+\,\mathcal{O}((\Delta t)^2))$$

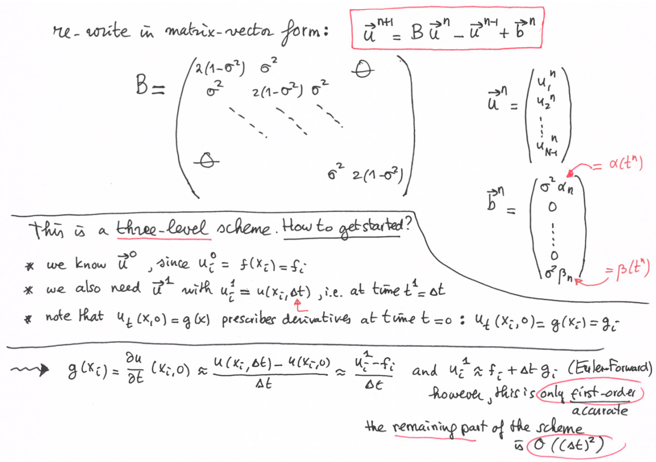
$$u_{xx}(x_i,t^n) \approx \frac{u(x_{i+1},t^n) - 2u(x_i,t^n) + u(x_{i-1},t^n)}{(\Delta x)^2} \quad (+\,\mathcal{O}((\Delta x)^2))$$

substitute in PDE

$\Rightarrow$ $\boxed{u_i^{n+1} = \sigma^2 u_{i+1}^n + 2(1-\sigma^2)u_i^n + \sigma^2 u_{i-1}^n - u_i^{n-1}}$ $i = ---$, $n = ---$

define $\boxed{\sigma = \frac{c\Delta t}{\Delta x}} > 0$

Wave equation [4]



re-write in matrix-vector form: $\boxed{\vec{u}^{\,n+1} = B\vec{u}^{\,n} - \vec{u}^{\,n-1} + \vec{b}^{\,n}}$

$$B = \begin{pmatrix} 2(1-\sigma^2) & \sigma^2 & & \Theta \\ \sigma^2 & 2(1-\sigma^2) & \sigma^2 & \\ & \ddots & \ddots & \ddots \\ \Theta & & \sigma^2 & 2(1-\sigma^2) \end{pmatrix} \qquad \vec{u}^{\,n} = \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N-1}^n \end{pmatrix}$$

$$\vec{b}^{\,n} = \begin{pmatrix} \sigma^2 \alpha_n \\ 0 \\ \vdots \\ 0 \\ \sigma^2 \beta_n \end{pmatrix} \qquad \alpha_n = \alpha(t^n), \quad \beta_n = \beta(t^n)$$

This is a three-level scheme. How to get started?

- we know $\vec{u}^{\,0}$, since $u_i^0 = f(x_i) = f_i$
- we also need $\vec{u}^{\,1}$ with $u_i^1 = u(x_i, \Delta t)$, i.e. at time $t^1 = \Delta t$
- note that $u_t(x,0) = g(x)$ prescribes derivatives at time $t=0$: $u_t(x_i,0) = g(x_i) = g_i$

$$\rightsquigarrow \; g(x_i) = \frac{\partial u}{\partial t}(x_i,0) \approx \frac{u(x_i,\Delta t) - u(x_i,0)}{\Delta t} \approx \frac{u_i^1 - f_i}{\Delta t}$$

and $u_i^1 \approx f_i + \Delta t\, g_i$ (Euler-Forward)

however, this is only first-order accurate

the remaining part of the scheme is $\mathcal{O}((\Delta t)^2)$

Wave equation [5]

Taylor:

$$\frac{u(x_i,\Delta t)-u(x_i,0)}{\Delta t} = \frac{\partial u}{\partial t}(x_i,0) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_i,0) + \mathcal{O}((\Delta t)^2)$$

$$= \frac{\partial u}{\partial t}(x_i,0) + \frac{c^2\Delta t}{2}\frac{\partial^2 u}{\partial x^2}(x_i,0) + \mathcal{O}((\Delta t)^2)$$

$$\rightsquigarrow \quad u_i^1 = u(x_i,\Delta t) \approx u(x_i,0) + \Delta t\,\frac{\partial u}{\partial t}(x_i,0) + \frac{c^2(\Delta t)^2}{2}\frac{\partial^2 u}{\partial x^2}(x_i,0)$$

$$\approx f(x_i) + \Delta t \cdot g(x_i) + \frac{c^2(\Delta t)^2}{2} f''(x_i,0)$$

$$\approx f_i + \Delta t \cdot g_i + \frac{c^2(\Delta t)^2}{2(\Delta x)^2}\left(f_{i+1} - 2f_i + f_{i-1}\right)$$

$$= \frac{1}{2}\sigma^2 f_{i+1} + (1-\sigma^2) f_i + \frac{1}{2}\sigma^2 f_{i-1} + \Delta t \cdot g_i$$

in matrix_vector form:

$$\begin{cases} \vec{u}^{\,0} = \vec{f} \\ \vec{u}^{\,1} = \frac{1}{2} B \vec{u}^{\,0} + \Delta t\, \vec{g} + \frac{1}{2} \vec{b}^{\,0} \end{cases}$$

$$\rightsquigarrow \quad \boxed{\mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta x)^2)}$$

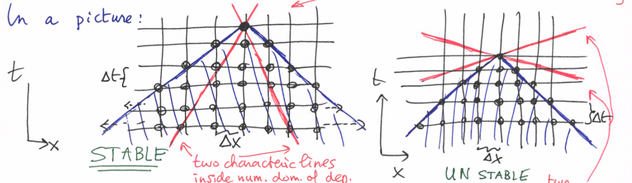
Wave equation [6]

Stability ("CFL-condition" for the wave equation)

"characteristics starting from grid point (x_i, t^n) must remain in the numerical domain of dependence for all $0 \le t \le t^n$"

here: triangle $\{(x,t) \mid 0 \le t \le t^n;\ x_i - t^n + t \le x \le x_i + t^n - t\}$

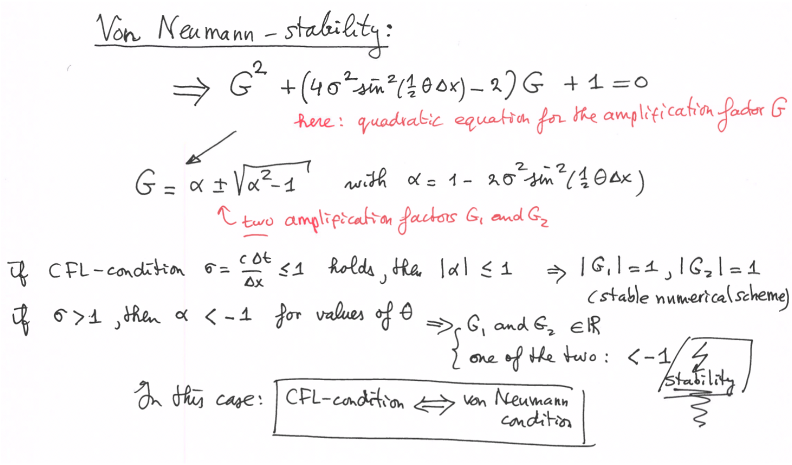
In a picture:



Characteristic lines are lines with slope ±c

$\Rightarrow$ CFL-condition: $\frac{c\Delta t}{\Delta x} \le 1$ (necessary not sufficient)

Wave equation [7]



Von Neumann-stability:

$$\Rightarrow G^2 + \left(4\sigma^2 \sin^2\left(\tfrac{1}{2}\theta\Delta x\right) - 2\right) G + 1 = 0$$

here: quadratic equation for the amplification factor G

$$G = \alpha \pm \sqrt{\alpha^2 - 1} \quad \text{with} \quad \alpha = 1 - 2\sigma^2 \sin^2\left(\tfrac{1}{2}\theta\Delta x\right)$$

two amplification factors G_1 and G_2

If CFL-condition $\sigma = \frac{c\Delta t}{\Delta x} \leq 1$ holds, then $|\alpha| \leq 1$ $\Rightarrow |G_1| = 1, |G_2| = 1$ (stable numerical scheme)

If $\sigma > 1$, then $\alpha < -1$ for values of θ $\Rightarrow \begin{cases} G_1 \text{ and } G_2 \in \mathbb{R} \\ \text{one of the two: } < -1 \end{cases}$ stability

In this case: $\boxed{\text{CFL-condition} \Longleftrightarrow \text{von Neumann condition}}$

Wave equation [8]

2D wave equation : $u_{tt} = c^2 \Delta u$

$$\sigma_x = \frac{c\Delta t}{\Delta x}$$

$$\sigma_y = \frac{c\Delta t}{\Delta y}$$

CTCS

$\Delta x = \Delta y = h$

stability criterion : $\boxed{\frac{c\,\Delta t}{h} \le \frac{1}{\sqrt{2}}}$

in d space dimensions: $\boxed{\frac{c\Delta t}{h} \le \frac{1}{\sqrt{d}}}$

Outlook to Lecture 7

- prepare exercises of Lecture 6 (see webpage!)
- Lecture 7: exact and nonstandard finite differences
- splitting and explicit-implicit methods (optional)
- exponential integrators (optional)