

# Numerical Methods for Time-Dependent PDEs

## Example questions for an Exam (Tentamen): WISL602

May 2025

⌚ The exam will be closed book.

📖 Review the workgroup exercises from lectures 1-9, 11 and 12, as well!

🕒 This mock exam is slightly longer to cover many topics.

### Question 1

#### METHOD OF UNDETERMINED COEFFICIENTS

(a) Use the *method of undetermined coefficients* to derive a *fourth-order* accurate central approximation of  $u_x$  at the grid point  $x_j = j\Delta x$ ,  $j = 0, 1, \dots, J$  with  $\Delta x = \frac{1}{J}$  of the form (find the constants  $A$  and  $B$ ):

$$\frac{-Au_{j+2} + Bu_{j+1} - Bu_{j-1} + Au_{j-2}}{\Delta x}.$$

(b) *Determine* the constants  $C$  and  $p$  in the error term:  $C(\Delta x)^4 u^{(p)}(\xi)$ .

### Question 2

#### TIME-INTEGRATION/LOCAL TRUNCATION ERROR/STABILITY

(a) Consider the following method for the ODE  $\dot{y} = f(t, y)$ :

$$y^{n+1} = y^n + \Delta t \left[ \frac{1}{7} f(t^{n+1}, y^{n+1}) + \frac{6}{7} f(t^n, y^n) \right].$$

Calculate the *local truncation error*  $\tau$  for this method.

(b) Derive the *stability function*<sup>1</sup>  $R(z)$  for the method in (a). Is the method *(un)conditionally stable*? Explain.

(c) Show that the *stability region* is bounded by a circle: *determine*  $\alpha$  and  $r$  in:

$$[\Re(z) + \alpha]^2 + [\Im(z)]^2 = r^2.$$

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<sup>1</sup>It is *convenient* to define  $z = \lambda \Delta t$  with  $z = \Re(z) + \Im(z)\mathbf{i}$  and  $\mathbf{i} = \sqrt{-1}$ .

### Question 3

BOUNDARY LOCUS/STABILITY POLYNOMIAL/ZERO-STABLE

Consider the ODE:  $\dot{y}(t) = f(y(t))$ .

(a) Give the definition of the *boundary locus* and explain why it plays an important role in the numerical solution of ODEs.

(b) Let the following *six* functions be defined for  $\varphi \in [0, 2\pi]$ :

F1.  $g_1(\varphi) = 1 - e^{-\mathbf{i}\varphi}$ , with  $\mathbf{i} = \sqrt{-1}$ .

F2.  $g_2(\varphi) = \frac{3}{2} - 2e^{-\mathbf{i}\varphi} + \frac{1}{2}e^{-2\mathbf{i}\varphi}$ .

F3.  $g_3(\varphi) = \mathbf{i} \sin(\varphi)$ .

F4.  $g_4(\varphi) = \frac{3\mathbf{i} \sin(\varphi)}{\cos(\varphi)+2}$ .

F5.  $g_5(\varphi) = 1 + e^{-\mathbf{i}\varphi} + \frac{1}{2}(1 - e^{-\mathbf{i}\varphi})^2 + \frac{1}{3}(1 - e^{-\mathbf{i}\varphi})^3$ .

F6.  $g_6(\varphi) = e^{\mathbf{i}\varphi} - 1$ .

and also the following *six* time-integration methods:

M1: Euler-Forward.

M2: BDF2 (an extension of Euler-Backward).

M3: Euler-Backward.

M4: explicit Midpoint.

M5: implicit Milne-Simpson:  $y^{n+2} = y^n + \Delta t [\frac{1}{3}f(y^n) + \frac{4}{3}f(y^{n+1}) + \frac{1}{3}f(y^{n+2})]$ .

M6: BDF3 (an extension of BDF2).

Associate each method M1, ..., M6 with one of the functions F1, ..., F6 and explain your choice. What do the functions  $g(\varphi) : [0, 2\pi] \rightarrow \mathbb{C}$  represent?

(c) Sketch the curves F1, F3 and F6 in the complex plane.

(d) Find the *stability polynomial*  $\pi(z; \zeta)$  for the 3-step Adams-Bashforth method:

$$y^{n+3} = y^{n+2} + \frac{\Delta t}{12}[5f(y^n) - 16f(y^{n+1}) + 23f(y^{n+2})].$$

(e) Consider the following two *linear multistep* methods:

$$3y^{n+2} = 4y^{n+1} - y^n + 2\Delta t f(y^n)$$

and

$$y^{n+2} = 4y^{n+1} - 3y^n - 2\Delta t f(y^n).$$

Verify whether these methods are *consistent* and *zero-stable*.<sup>2</sup>

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<sup>2</sup>Hint: use properties of the *characteristic polynomials*:  $\rho(\zeta)$  and  $\sigma(\zeta)$ .

## Question 4

### HEAT/ADVECTION

Consider the *advection* equation:

$$u_t + \gamma u_x = 0, \quad u(x, 0) = e^{-x^2}, \quad x \in (-\infty, \infty), \quad \gamma \in \mathbb{R}.$$

- (a) What is the *exact* solution  $u(x, t)$  of this model?
- (b) Show, using *Von Neumann* stability analysis, that the numerical scheme

$$u_j^{n+1} = (1 + \frac{\gamma \Delta t}{\Delta x}) u_j^n - \frac{\gamma \Delta t}{\Delta x} u_{j+1}^n$$

is always *unstable* for  $\gamma = 1$  and *conditionally* stable for  $\gamma = -1$ .

- (c) The method in (b) is *first-order accurate*. What happens for  $\frac{\gamma \Delta t}{\Delta x} = -1$ ?

## Question 5

### FINITE ELEMENTS

Apply a *finite element method* with piecewise linear functions  $\phi_j$ , both as approximating and test functions, to the PDE:  $u_t = u_{xx} + f(t)$ .

- Show that a *semi-discrete* ODE-system is obtained of the form:

$$\mathcal{A} \dot{\vec{u}}(t) + \mathcal{B} \vec{u}(t) = \vec{f}.$$

- Describe the *inner products*  $\langle \cdot, \cdot \rangle$  appearing in the matrices  $\mathcal{A}$  and  $\mathcal{B}$ .
- Which *names* are usually associated with the matrices  $\mathcal{A}$  and  $\mathcal{B}$ ?

## Question 6

### ADAPTIVE/NON-UNIFORM GRIDS

The *equidistribution* principle with monitor function  $\omega > 0$  reads:

$$[x_\xi \omega]_\xi = 0, \quad x(0) = 0, \quad x(1) = 1. \quad (1)$$

A *non-uniform grid* in the variable  $x$  is determined via the grid function  $x(\xi)$  on a uniform grid distribution for  $\xi$ .

- (a) Calculate the grid function  $x(\xi)$  for  $\omega = 1$ . Which grid distribution follows from (1) in this case?
- (b) Which property for the derivative of the transformation  $\xi \rightarrow x$  ensures a *non-distorted* (= non-singular) non-uniform grid, i.e., a grid with  $\Delta x_j > 0, \forall j$ .

(c) Let the transformation  $x(\xi) = 4\xi^3 + \xi^2 - 4\xi$  be given. Make clear that the property in part (b) is violated.

(d) Next, consider  $x(\xi) = \xi^4$ . Check that the property in part (b) is satisfied for this grid function.

(e) Define the *five* grid points  $\xi_j = j\Delta\xi$ ,  $j = 0, 1, \dots, 4$ . Sketch  $x$  from parts (c) and (d) as a function of  $\xi \in [0, 1]$  and explain how you obtain the values of the non-uniform grid points  $x_j$  from these graphs (and mark those points).

## Question 7

EXACT/NONSTANDARD FDs

(a) Verify that the scheme:

$$\left\{ \begin{array}{l} \frac{y^{n+1} - y^n}{1 - e^{-\Delta t}} = -y^n, \quad n = 0, 1, 2, \dots; \quad \Delta t > 0, \\ y^0 = 1, \quad \text{with } y^n \approx y(t^n) = y(n\Delta t) \end{array} \right\}$$

is an *exact* finite difference (FD) scheme for the ODE:  $\left\{ \begin{array}{l} \dot{y}(t) = -y(t), \\ y(0) = 1. \end{array} \right\}$ .

(b1) Which *denominator* function  $\phi$  is being used in this case?

(b2) What is the essential property of  $\phi$  in exact and nonstandard FDs?

(b3) Which  $\phi$  is related to the Euler-Forward method?

(b4) Is the above scheme a *nonlocal* one? Explain!

## Question 8

HAMILTONIAN PDE

Consider the time-dependent PDE:  $u_t = u^q u_x - \epsilon u_{xxx}$  with  $q \geq 2$ ,  $\epsilon > 0$  and  $u(x, t) \rightarrow 0$  as  $|x| \rightarrow +\infty$ .

(a) Calculate the *variational* derivative of the functional:

$$\mathcal{H} = \int_{-\infty}^{\infty} \left[ \frac{\epsilon}{2} u_x^2 + \frac{1}{(q+1)(q+2)} u^{q+2} \right] dx.$$

(b) Show that the PDE is a *Hamiltonian* PDE.

(c) Briefly<sup>3</sup> describe a *discretization* on the finite spatial interval  $[-L, L]$  with  $L \gg 1$  and a useful *time-integration* method for the Hamiltonian PDE in (b).

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<sup>3</sup>For this answer, use (at most) three lines!

## Question 9

### TRAVELLING WAVES

Please check the exercises on TWs in lecture 11.

## Question 10

### CLASSIFICATION/METHOD OF LINES/TIME-INTEGRATION/STABILITY

Consider the *Telegraph* equation:

$$u_{tt} + \alpha u_t + \beta u = c^2 u_{xx}, \quad \alpha \in \mathbb{R}, \quad \beta \in \mathbb{R}, \quad c \in \mathbb{R}. \quad (2)$$

(a) *Classify* PDE (2): is it elliptic, parabolic or hyperbolic?

(b) *Re-write* the PDE as a system of first-order PDEs in time, semi-discretize in space (the *first step* in the Method-of-Lines) and write the resulting ODE system<sup>4</sup> as:

$$\dot{\vec{\eta}} = M \vec{\eta}. \quad (3)$$

and *verify* that:

$$M = \begin{pmatrix} \mathcal{O} & \mathcal{I} \\ c^2 D_{2c} - \beta \mathcal{I} & -\alpha \mathcal{I} \end{pmatrix} \quad \text{with} \quad \vec{\eta} := [\vec{U}, \vec{V}]^T.$$

It can be derived that the eigenvalues  $\mu$  of the matrix  $M$  satisfy<sup>5</sup>:

$$\mu^2 + \alpha\mu + \beta - c^2\lambda = 0,$$

where  $\lambda$  is an eigenvalue<sup>6</sup> of the matrix  $D_{2c}$ .

Next, consider in the *second step* of the Method-of-Lines, the following four time-integration methods:

1. Euler-Forward (EF).
2. Euler-Backward (EB).
3. (explicit) Midpoint method (MP).
4. Boundary-Value method (MP as underlying method with EB final condition).

(c) For the case  $\alpha = \beta = 0$ ,  $c = 1$ : which methods would you recommend regarding their *stability properties* and which ones not? Motivate your choice!

(d) Same question as in part (c), but now for the case:  $\alpha = 2$ ,  $\beta = 0$ ,  $c = 2$ .

(e) Same question as in part (c) for the case:  $\alpha = c = 0$ ,  $\beta = -100$

<sup>4</sup>As you have learned during the lectures: the second-order three-point central approximation for  $u_{xx}$  at the grid point  $x_p$  can be represented by the matrix  $D_{2c}$ .

<sup>5</sup>You do not have to prove this.

<sup>6</sup>The eigenvalues of the matrix  $D_{2c}$  are:  $\lambda_p = \frac{2}{(\Delta x)^2}(\cos(p\pi\Delta x) - 1)$ ,  $p = 1, 2, \dots, M$ .