Assignment C1A

(5 % of Final Grade)

February 2025

Deadline: see the webpage of the course.
(This is an individual assignment!)

PART A (25 points)

We are looking for a solution of the *backward* wave equation:

$$u_{tt} = -u_{xx},$$

with initial conditions $u(x,0) = \sin(2\pi x)$ and $u_t(x,0) = 0$, and with periodic boundary conditions $u(0,t) = u(1,t)$ and $u_x(0,t) = u_x(1,t)$.

The solution to this equation can be written in the form of a Fourier series:

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\sin(2\pi kx) + \sum_{k=0}^{\infty} b_k(t)\cos(2\pi kx).$$

(a) Show that $a_k(t) = b_k(t) = 0$ for all $k \neq 1$ and that:

$$\begin{cases} \ddot{a}_1(t) - 4\pi^2 a_1(t) = 0, \\ \ddot{b}_1(t) - 4\pi^2 b_1(t) = 0. \end{cases}$$

with $a_1(0) = 1$ and $a_1'(0) = b_1(0) = b_1'(0) = 0$.

(b) Conclude that the solution can be written as $u(x,t) = \cosh(2\pi t)\sin(2\pi x)$.

PART B (50 points)

Consider the one-dimensional *stationary* convection-diffusion model:

$$\begin{cases} -\delta u''(x) + \nu u'(x) = 0, \ x \in [0,1], \ \delta, \nu > 0, \\ u(0) = 0, \quad u(1) = 1. \end{cases} \tag{1}$$

The exact solution reads: $u(x) = \left(\exp\left(\frac{\nu}{\delta}x\right) - 1\right) / \left(\exp\left(\frac{\nu}{\delta}\right) - 1\right)$.

(a) Work out 1) a central difference scheme (both for u' and u'') and 2) a one-sided difference[1] scheme (for u', but still central for u'') in equation (1).

[1] Approximation: $u'|_{x_i} \approx \frac{u_i - u_{i-1}}{\Delta x}$.

Write the fully-discretized $(I-1)\times(I-1)$ linear system as: $\mathcal{A}\vec{u}=\vec{f}$, where

$$\mathcal{A}=\frac{1}{(\Delta x)^2}\begin{pmatrix} b & c & 0 & & & & & \\ a & b & c & 0 & & & \emptyset & \\ 0 & a & b & c & 0 & & & \\ & & \cdots & \cdots & \cdots & & & \\ & & & \cdots & \cdots & \cdots & & \\ & & & 0 & a & b & c & 0 \\ & \emptyset & & & 0 & a & b & c \\ & & & & & 0 & a & b \end{pmatrix},$$

and

$$\vec{u}=[u_1,u_2,...,u_{I-2},u_{I-1}]^T,\ \vec{f}=[f_1,0,...,0,f_{I-1}]^T.$$

Find the values a, b, c, f_1 and f_{I-1} for both schemes.

(b) Solve the following linear difference equations:

$$a\ u_{i-1}+b\ u_i+c\ u_{i+1}=0,\quad i=1,...,I-1,$$

$$\text{where } a<0,\ b>0,\ a+b+c=0,\ \tfrac{a}{c}\neq\pm1,\ c\neq 0$$

by using a trial solution of the form: $u_i=\alpha+\beta r^i$ with $u_0=0,\ u_I=1$.

(c) Consider the two cases $c>0$ and $c<0$. Check for the solutions in part (b), whether we can expect *positivity-preserving* numerical solutions ($u_i\geq 0$) and/or whether we can expect solutions with *non-physical oscillations* in the numerical values u_i. Show that neither of the difference schemes of (a) produces *oscillations* when $\Delta x<2\frac{\delta}{\nu}$ holds.

(d) Show that the (local) truncation error for the one-sided scheme is $\mathcal{O}(\Delta x)$ (for the original model), but $\mathcal{O}((\Delta x)^2)$ for the so-called <u>modified</u> equation:

$$-(\delta+\frac{\nu\Delta x}{2})u''(x)+\nu u'(x)=0,$$

where $\frac{\nu\Delta x}{2}$ is a numerical (artificial) diffusion coefficient.

(e) Plot a few relevant numerical solutions (choose specific values for ν and δ) to support the results from parts (a)-(d). Also compare the results with the exact solution of the model. Discuss the numerical solutions in terms of (unphysical) oscillations and (extra) damping.

See next page!

PART C (25 points)

(a) Construct a time-integration method for $y'(t) = f(y(t))$ that is consistent, but not *zero-stable*[2].

(b) Determine and plot the stability region in the complex plane of the following linear multistep method:

$$y^{n+2} - y^{n+1} = \Delta t f(y^n).$$

[2] It is not allowed to use the examples that are discussed in the exercises or lecture notes.