Numerical Methods for PDEs

Fall 2026

Exercises for Lecture 10

Exercise 10.1 (TWs)

Consider the Fisher equation from population dynamics ($u > 0$):

$$u_t = u_{xx} + \gamma u(1-u), \quad -\infty < x < \infty$$

with $\gamma > 0$ and boundary conditions $u(-\infty, t) = 1$ and $u(+\infty, t) = 0$. Determine the travelling wave ODE, analyze the phase plane and sketch the corresponding TW solution of the PDE.

Exercise 10.2 (TWs)

The same question as in Exercise 10.1, but now for the non-equilibrium geohydrology model from porous media ($u > 0$):

$$u_t = u_{xx} + [u^2]_x + \tau u_{xxt}, \quad -\infty < x < \infty, \quad \tau > 0$$

with boundary conditions $u(-\infty, t) = u_-$ and $u(+\infty, t) = u_+$ for $0 < u_- < u_+$.

Exercise 10.3 (TWs)

As in exercise 10.1, but now for the Fisher PDE with density-dependent diffusion ($u > 0$):

$$u_t = [uu_x]_x + u(1-u), \quad -\infty < x < \infty$$

with boundary conditions $u(-\infty, t) = 1$ and $u(+\infty, t) = 0$. Consider monotonically decreasing solutions. *Hint: You may assume without proof that the trivial equilibrium is a stable node.*

Exercise 10.4 (AMs)

Given the equidistribution principle

$$[x_\xi \omega]_\xi = 0, \quad x(0) = 0, \; x(1) = 1.$$

(a) Show that for the arclength-type monitor function $\omega = \sqrt{1 + \alpha u_x^2}$ we obtain a uniform grid distribution, if $\alpha \to 0$.

(b) What happens with the grid distribution for the monitor function $\omega = u_x$, if the solution has flat parts $u_x \approx 0$? Is such a distribution useful?

(c) Show that the equidistribution principle is obtained by minimizing the "grid energy" functional $E = \frac{1}{2}\int_0^1 \omega x_\xi^2 \, d\xi$. *Hint: Minimize this functional by employing Euler-Lagrange equations.*

Exercise 10.5 (AMs)

(a) Work out a general *non-uniform grid* approximation for u_x at the grid point x_i, only making use of the values u_{i-1}, u_i and u_{i+1} at x_{i-1}, x_i and x_{i+1}, respectively. Show at least three different approximations $u_x|_i \approx Au_{i-1}+Bu_i+Cu_{i+1}$.

(b) Derive the central difference approximation (as a special case from part ii): $u_x|_i \approx \frac{u_{i+1}-u_{i-1}}{x_{i+1}-x_{i-1}}$, work out its local truncation error τ in terms of the transformation derivatives x_ξ, $x_{\xi\xi}$, ... and solution derivatives u_{xx}, u_{xxx}, ...:

$$\tau = \epsilon H^2 + \mathcal{O}(H^3),$$

where $H := \Delta\xi$, the constant stepsize in the transformed variable ξ (find the factor ϵ).

(c) Considering only u_x as in part (b): which monitor function ω of the form $[\frac{\partial^p u}{\partial x^p}]^q$ in the equidistribution relation

$$[\omega x_\xi]_\xi = 0$$

yields '*supra-convergence*' on the non-uniform grid x_i? In other words, set $\epsilon = 0$ in part (b) and find the appropriate ω, for which $\tau = \mathcal{O}(H^2) \Rightarrow \tau = \mathcal{O}(H^3)$.

Exercise 10.6 (AMs)

Consider the heat equation $u_t = u_{xx}$.

(a) Apply a coordinate transformation of the form $x(\xi, \theta)$, $t = \theta$ and work out the transformed PDE.

(b) Suppose we want to use an equidistribution principle such as in Exercise 10.4. Which monitor function ω must be used to cancel the first term in the local truncation error of u_{xx} on a non-uniform grid? This is called *supra-convergence*. *Hint: follow a step plan similar to Exercise 10.5.*

Exercise 10.7 (AMs)

Consider the hyperbolic PDE $u_t + c(x)u_x = f(u)$. Apply a general coordinate transformation of the form

$$x = x(\xi, \theta), \quad t = \theta$$

and derive the transformed advection PDE:

$$u_\theta + \beta x_\theta = \gamma u_\xi + g.$$

Specify the functions β, γ and g. Also, give the Jacobian matrix (and its determinant) of this transformation.

Exercise 10.8 (AMs)

Consider the time-dependent grid transformation

$$\begin{aligned} x &= x(\xi, \eta, \theta), \\ y &= y(\xi, \eta, \theta), \\ t &= \theta. \end{aligned}$$

Show that the determinant of this transformation is given by:

$$J := \det(\mathcal{J}) = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}.$$

Also, determine the inverse $(\mathcal{J})^{-1}$.