Numerical Methods for PDEs

Fall 2026

Exercises for Lecture 12

FRACTIONAL PDEs

Exercise 12.1

Show that the (left) Caputo fractional derivative of order α ($1 < \alpha \leq 2$) is "consistent" with the traditional integer derivative of order 2, i.e. take the limit for the fractional order derivative $\alpha \in \mathbb{R} \to 2$ and show that we indeed obtain the integer derivative of order 2.

Exercise 12.2

Consider the (left- or right-) space-fractional time-dependent PDE (in Caputo sense)

$$u_t = D_C^\alpha u, \quad 1 \leq \alpha \leq 2, \quad x \in \mathbb{R},$$

with initial solution $u(x,0) = \mathrm{e}^{-x^2}$. Discuss the solution behaviour for $\lim_{\alpha\to 1}$, $\lim_{\alpha\to 2}$ and intermediate values of α. Also, describe the two cases: "$\int_{-\infty}^{x}$" (left derivative) and "$\int_{x}^{\infty}$" (right derivative).
How do we obtain a symmetric fractional diffusion behaviour? In the latter case, what is the difference with ordinary diffusion and what happens for the value $\alpha = 1$?

Exercise 12.3

(a) Check that the solution of the fractional ODE (not imposing any initial condition!):

$$\mathcal{D}_C^{\frac{3}{2}} u(t) = \frac{\Gamma(6) t^{\frac{7}{2}}}{\Gamma(\frac{9}{2})}$$

is given by: $u(t) = t^5$. (in fact, it can be proved that it is the only *analytic* solution of this ODE!)

(b) [*] The solution of the space-fractional PDE with $1 < \alpha \leq 2$

$$\begin{cases} u_t = -(-\Delta u)^{\frac{\alpha}{2}}, & x \in [0,1], \\ u_{|t=0} = u_0(x), \end{cases}$$

with homogeneous boundary conditions at $x = 0$ and $x = 1$ is given by the following Fourier-type series:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \mathrm{e}^{-n^\alpha \pi^\alpha t},$$

where $c_n = 2\int_0^1 u_0(s)\sin(n\pi s)\, ds$. Plot, in one figure, the solution for $\alpha = 2$ and $\alpha = 3/2$ at $t = 0$ and $t = 0.1$. For this, choose an appropriate final index value $M \in \mathbb{N} \ll 1$ in the series.

Exercise 12.4

Derive the following first-order finite-difference approximation for $D_C^{3/2}u$:

$$D_C^{3/2}u_{|x_i} \approx \frac{1}{\Gamma(3/2)(\Delta x)^{3/2}} \sum_{j=1}^{i-1} \{(j+1)^{1/2} - j^{1/2}\}\{u_{i-j+1} - 2u_{i-j} + u_{i-j-1}\}$$

Describe the structure (zeroes and non-zeroes) of the underlying finite-difference matrix (with homogeneous Neumann boundary conditions).

Exercise 12.5

(a) Check that $D_C^\alpha(c) = 0$ and find $D_{RL}^\alpha(c)$, where $c \in \mathbb{R}$.

(b) Show that the fractional derivative (in terms of Caputo or Riemann-Liouville) is a linear operator.

(c) Check that $f(x) = \cos(2m\pi x)\ \Gamma(x)$ $(m \in \mathbb{N}^{\geq 0})$ solves the functional equation:

$$\begin{cases} f(x+1) = xf(x), & x > 0, \\ f(1) = 1. \end{cases}$$

(d) Plot the function $f(x)$ in part (c) for $m = 0, 1, 2$ and explain why $m = 0$ is the only value for practical use in fractional calculus.

(e) Calculate the values $\Gamma(\frac{1}{2})$, $\Gamma(\frac{3}{2}))$, $\Gamma(\frac{5}{2})$, ...

(f) Plot the function $E_\alpha(-x^\alpha)$ for $\alpha = 1, \frac{3}{4}, \frac{1}{2}$ and $\frac{1}{4}$, where $E_\alpha(x)$ $(x > 0)$ is the Mittag-Leffler function.

Exercise 12.6 [*]

Consider the (left) space-fractional advection-diffusion (dispersion) PDE[1]:

$$\frac{\partial u(x,t)}{\partial t} = \epsilon \; D_C^{\alpha} u(x,t) - \gamma \frac{\partial u(x,t)}{\partial x}, \quad x_L < x < x_R, \epsilon > 0, \gamma > 0.$$

(a) Show that Euler-Forward combined with the Grünwald approximation defined by equation (3) in the mentioned extra file, applied to the advection-diffusion (dispersion) equation, is unstable.

(b) Similar question for Euler-Backward combined with the Grünwald approximation defined by equation (3) in the extra file: it is unstable as well!

(c) Show that the shifted Grünwald approximation defined by equation (10) in the extra file is consistent with the Riemann-Liouville fractional derivative of equation, defined in equation (2).

(d) Show that the shifted Grünwald approximation (10), applied to the advection-diffusion (dispersion) equation **is** unconditionally *stable*.

BOUNDARY-VALUE METHODS

Exercise 12.7

(a) Consider the ODE:

$$\begin{cases} u'(t) = \lambda \; u(t), \quad \lambda \in \mathbb{C}, \\ u(0) = u_0. \end{cases}$$

Work out the system of linear equations that is obtained when the following BV-method is applied:

1) Euler-Forward in the first time-step
2) Explicit-midpoint for the intermediate time-steps
3) Euler-Backward for the final time-step

As an alternative to Euler-Backward (EB), use BDF2 in the final step. Why could be this be advantageous? Also, comment on the special case: $\lambda = i = \sqrt{-1}$.

(b) Find the exact numerical solution $u_n, \; n \in \mathbb{N}$, of the linear system in (a). Comment on the advantages of the BV-method for $Re(\lambda) > 0$.

[1] Check the article by Meerschaert and Tadjeran on the webpage = one of the extra files.

(c) The same question as in part (a) but now for the linear ODE system:

$$\begin{cases} \vec{u}'(t) = \mathcal{A}\,\vec{u}(t), \quad \mathcal{A} \in \mathbb{R}^{2\times 2}, \\ \vec{u}(0) = \vec{u}_0. \end{cases}$$

Suppose that all eigenvalues of the matrix $\mathcal{A}$ are real and positive. Comment on the applicability (in terms of numerical stability) of traditional explicit and implicit time-integrators.

(d) Apply the BV-method from (a) to the nonlinear ODE:

$$\begin{cases} u'(t) = f(u(t)), \\ u(0) = u_0 \end{cases}$$

and describe the nonlinear system to be solved.

(e) Solve the ODE in (d) analytically for the cases: $f(u) = u^2$ and $f(u) = \mathrm{e}^{\alpha u}$, $\alpha \in \mathbb{R}$. What additional complications now arise for *both* traditional time-integration methods *and* BV-methods? How could this be resolved numerically? (also check Lecture 10!)

Exercise 12.8

Show that the *boundary locus* of any *consistent* linear multistep method possesses the following properties:

$\Vdash$ the origin in the complex plane is always on the boundary locus.

$\Vdash$ the boundary locus is symmetric with respect to the real axis in $\mathbb{C}$.

$\Vdash$ at the origin, it is perpendicular to the real axis.

Moreover, show that the stability region of the (lowest-order) midpoint BV-method, as discussed in Lecture 12, is the whole complex plane $\mathbb{C}$, excluding the imaginary axis $i\mathbb{R}$.

Exercise 12.9

(a) Apply the *doubling-splitting* procedure (see the lecture notes of Lecture 12) for the half-fractional PDE:

$$\begin{cases} u_t = -(-\Delta u)^{\frac{1}{2}}, \quad x \in [-L, L],\ L \gg 1, \\ u_{|t=0} = u_0(x), \quad \text{with } \lim_{x\to\pm\infty} u_0(x) = 0. \end{cases}$$

(b) Describe the method-of-lines and the resulting ODE system for this PDE. Comment on the eigenvalues of the matrix and the consequences/choices for the

time-integration method.

(c) Without performing the doubling-splitting procedure to this PDE, what could be an alternative to do the first step in the method of lines (in terms of square roots of matrices)?

Exercise 12.10

The same questions (a), (b), (c) as in exercise 12.9, but now for the *left*-space fractional heat equation of order 8/7:

$$\begin{cases} u_t = \mathcal{D}_C^{\frac{8}{7}} u, \quad x \in [-L, L], \ L \gg 1, \\ u_{|t=0} = u_0(x), \quad \text{with } \lim_{x \to \pm\infty} u_0(x) = 0. \end{cases}$$

An additional question for this PDE model: what changes in the doubling-splitting step in case of the *right*-space fractional derivative?