Numerical Methods for PDEs

Exercises for Lecture 2

Exercise 2.1: Construction of the Central Difference Matrix

Consider a uniform grid with spacing h and the shift matrices S_+ and S_-.

1. Show that the central difference approximation

$$u'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h}$$

   can be written as a matrix operator.

2. Express the resulting matrix in terms of

$$S_+, \qquad S_-.$$

3. Construct the explicit matrix for $n = 6$.

Exercise 2.2: Construction of the Discrete Second Derivative

The second derivative is approximated by the central difference formula

$$u''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

1. Derive the matrix representation using the shift matrices.
2. Write the resulting matrix explicitly for $n = 5$.
3. Explain why this matrix is tridiagonal.

Exercise 2.3: Symmetry Properties of Difference Matrices

Consider the matrices

$$D_{1,c} = \frac{1}{2h}(S_+ - S_-),$$

and

$$D_2 = \frac{1}{h^2}(S_+ - 2I + S_-).$$

1. Prove that

$$D_{1,c}^T = -D_{1,c}.$$

2. Prove that

$$D_2^T = D_2.$$

3. Explain why the symmetry properties of these matrices are important in numerical analysis.

Exercise 2.4: **Higher-Order Shift Operators**

Consider the second forward difference

$$\Delta^2 u_i = u_{i+2} - 2u_{i+1} + u_i.$$

1. Express this operator using powers of the forward shift matrix.
2. Write the corresponding matrix expression.
3. Compare this operator with the central second derivative matrix

$$D_2 = \frac{1}{h^2}(S_+ - 2I + S_-).$$

Discuss the difference between the two approximations.

Exercise 2.5: **General Three-Point Stencils**

Consider a general three-point finite-difference operator

$$(Au)_i = a_{-1}u_{i-1} + a_0 u_i + a_1 u_{i+1}.$$

1. Prove that this operator can be written as

$$A = a_{-1}S_- + a_0 I + a_1 S_+.$$

2. Use this representation to construct a matrix approximation of

$$u'' + 3u'.$$

Use the approximations

$$u'' \approx D_2 u,$$

and

$$u' \approx D_{1,c}u.$$

3. Write the final matrix operator explicitly in terms of

$$S_+, \qquad S_-, \qquad I.$$

Exercise 2.6: General Finite Stencils

Consider a finite-difference operator with stencil width m,

$$(Au)_i = \sum_{k=-m}^{m} a_k u_{i+k}.$$

1. Show that this operator can be written as

$$A = \sum_{k=-m}^{m} a_k S^{(k)}.$$

2. Explain why the resulting matrix has bandwidth

$$2m + 1.$$

3. Discuss why sparse matrix structure is important for solving large systems arising from discretized PDEs.

Exercise 2.7: Method of undetermined coefficients

(a) Use the *method of undetermined coefficients* to obtain a uniform grid FD approximation of u_x at $x = x_i$ of the form

$$u_{x,i} \approx Au_{i+2} + Bu_{i+1} + Cu_i + Du_{i-1},$$

with a leading term in the error that is $\mathcal{O}((\Delta x)^3)$.

(b) Construct, for the first-derivative operator u_x at the gridpoint $x = x_i$, an approximation of the form:

$$u_{x,i} \approx Au_{i+1} + Bu_i + Cu_{i-1}.$$

How many solutions exist? Discuss a few well-known choices and determine the corresponding leading order errors.

Exercise 2.8: **Eigenvalues of FD matrices**

(a) Show that the eigenvalues of the (tri-diagonal) $M \times M$-matrix $\mathcal{D}$ (with *homogeneous Dirichlet* boundary conditions), given by

$$\mathcal{D} = \frac{1}{(\Delta x)^2} \begin{pmatrix} a & b & 0 & & & & & \\ b & a & b & 0 & & & \emptyset & \\ 0 & b & a & b & 0 & & & \\ & & \ldots & \ldots & \ldots & & & \\ & & & \ldots & \ldots & \ldots & & \\ & & & 0 & b & a & b & 0 \\ & \emptyset & & & 0 & b & a & b \\ & & & & & 0 & b & a \end{pmatrix},$$

are:

$$\lambda_p = \frac{1}{(\Delta x)^2}(a + 2b\cos(\frac{p\pi}{M+1})), \quad \text{for } p = 1, 2, ..., M.$$

(b) Pick $a = -2$, $b = 1$ and $\Delta x = 1$. Choose $M = 10$ and plot the eigenvalues in the complex plane. The same question for $M = 20$. What do you expect for $M \gg 1$?

Exercise 2.9: **Relationships between FD matrices**

(a) Use Matlab to verify that the following two identities hold:

$$\mathcal{D}_{2c} = \mathcal{D}_{1+}\mathcal{D}_{1-} \quad \& \quad \mathcal{D}_{1c} = \frac{1}{2}(\mathcal{D}_{1+} + \mathcal{D}_{1-}).$$

(b) Use Matlab to work out the matrix: $\mathcal{D}_{4c} = (\mathcal{D}_{2c})^2$.
(the matrix $\mathcal{D}_{4c}$ approximates $u_{4x} := u_{xxxx}$)

(c) Derive a finite difference matrix $\mathcal{D}_{6c}$ for $u_{6x} := u_{xxxxxx}$ by using Matlab.

Hint: Save your Matlab files of this exercise. They could come in handy later on in the course.

Exercise 2.10: **Higher dimensions**

In two dimensions the Laplacian Δu can be approximated at the gridpoint $(x, y) = (x_{i,j}, y_{i,j})$ on a *nine-point* stencil (take $\Delta x = \Delta y := h$):

$$\Delta u_{i,j} \approx \tfrac{1}{6h^2}[4(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}) + u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 20u_{i,j}].$$

Check that this approximation is of second order (similar to the five point central approximation), but becomes of higher order, when the Laplace equation $\Delta u = 0$ is solved, or when the Poisson equation $\Delta u = f(x, y)$ is solved with a harmonic function f, i.e., $\Delta f = 0$.