Numerical Methods for PDEs

Exercises for Lecture 3

Exercise 3.1

Show that

(a) the implicit Gear method

$$u^{n+1} = \frac{1}{3}(4u^n - u^{n-1}) + \frac{2\Delta t}{3} f(u^{n+1})$$

is *zero stable*,

(b) the explicit 3-step Adams method

$$u^{n+3} = u^{n+2} + \frac{\Delta t}{12}[5f(u^n) - 16f(u^{n+1}) + 23f(u^{n+2})]$$

is *zero stable*,

(c) and that the linear multistep method

$$u^{n+2} - 3u^{n+1} + 2u^n = -\Delta t f(u^n)$$

is <u>not</u> *zero stable*.

Exercise 3.2

Find the *stability polynomial* $\pi(\zeta; z)$ and its roots for the

(a) trapezoidal method:

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}[f(u^n) + f(u^{n+1})].$$

(b) the midpoint (leapfrog) method:

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = f(u^n).$$

Exercise 3.3

Sketch/plot[1] *the stability regions* in the complex plane of the following methods:

(a) *a* second-order explicit Runge-Kutta (two-stages) of the form:

$$\left\{ \begin{array}{l} k_1 = f(u^n) \\ k_2 = f(u^n + k_1 \Delta t), \\ u^{n+1} = u^n + \frac{\Delta t}{2}(k_1 + k_2). \end{array} \right\}$$

(b) *a* fourth-order explicit Runge-Kutta (four stages):

$$\left\{ \begin{array}{l} k_1 = f(u^n), \\ k_2 = f(u^n + k_1 \frac{\Delta t}{2}), \\ k_3 = f(u^n + k_2 \frac{\Delta t}{2}), \\ k_4 = f(u^n + k_3 \Delta t), \\ u^{n+1} = u^n + \frac{\Delta t}{6}[k_1 + 2k_2 + 2k_3 + k_4]. \end{array} \right\}$$

(c) *the* second-order Taylor method:

$$u^{n+1} = u^n + \Delta t f(u^n) + \frac{1}{2}(\Delta t)^2 f'(u^n) f(u^n).$$

Exercise 3.4

Show that the *boundary locus* of the method

$$u^{n+1} = u^n + \Delta t f(u^n + \Delta t f(u^n))$$

is defined by

$$[1 + x + x^2 - y^2]^2 + y^2[1 + 2x]^2 = 1.$$

Plot the corresponding curve.

Exercise 3.5

(a) Using Matlab, compute the eigenvalues of the 10×10 central finite-difference matrices $\mathcal{D}_{1c}$, $\mathcal{D}_{2c}$, $\mathcal{D}_{3c}$, $\mathcal{D}_{4c}$ and $\mathcal{D}_{6c}$. Then, plot the eigenvalues in the complex plane to visualize their positions.

(b) Comment on the stability properties of the methods EF (explicit Euler) and EB (implicit Euler), when applied to the semi-discrete ODE system:

$$\dot{\vec{u}} = \mathcal{D}_{mc}\vec{u}, \quad (m = 1, 2, 3, 4, 6).$$

[1] write $z = x + iy$ and find an approximation of the stability region and the boundary locus for a finite set of (x, y)-values.

Exercise 3.6

Work out the semi-discrete ODE system (applying the Method-of-Lines) for the following PDE models:

(a) $u_t = du_{xx} + (u^2)_x - \mu u_{xxt}$.

(b) $u_t = u_{xxx} + 6uu_x$.