approximate the fractional Laplacian for $\alpha \in (0,1)$ by,

$$-(-\Delta)^{\alpha/2}u(x)\Big|_{x=x_n} \approx -\frac{1}{2\cos(\alpha\pi/2)}\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\left(\sum_{k=0}^{n-1}\left((n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right)\left(u(x_{k+1})-u(x_k)\right)\right.$$
$$\left.-\sum_{k=n}^{N}\left((k+1-n)^{1-\alpha}-(k-n)^{1-\alpha}\right)\left(u(x_{k+1})-u(x_k)\right)\right). \tag{2.47}$$

This L1 method for the left derivative has error estimate given by [34] [31] [36] [50],

$$\left|\frac{u(a)h^{\alpha}}{\Gamma(1-\alpha)}+\sum_{k=0}^{n-1}\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\left((k+1)^{1-\alpha}-k^{1-\alpha}\right)\left(u(x_{k+1})-u(x_k)\right)-{}_{RL}D^{\alpha}_{a,x}u(x)\Big|_{x=x_n}\right| \leq Ch^{2-\alpha}, \tag{2.48}$$

for some constant C, so we can see that the method has lower error for small α. The left approximation has already been extended to work on non-uniform grids, so this make this approach a good candidate for finding a non-uniform grid approximation to the fractional Laplacian.

2.3.2 L2 method

This is a method from [34], commonly used and adapted in the literature [38] [1], which approximates the left Caputo derivative for $\alpha \in (1,2)$ on an interval $x \in [0,b]$ on a uniform grid. In the limit $\alpha \to 2$ the method reduces to the central 3-point approximation for the second derivative. The accuracy of the method is dependent on the chosen value for α and is most accurate for $\alpha \in (1,1.5)$. There exists a slightly adapted method known as L2C which performs better for $\alpha \in (1.5,2)$ [34], for simplicity we will only consider the L2 method in this thesis. Here, we adapt the L2 method to work over a general interval $x \in [a,b]$ where $a,b \in \mathbb{R}$ and $a<b$. We also use parallel reasoning to find an approximation of the right Caputo derivative on the same uniform grid and interval with $N+2$ grid points as used in the L1 method.

The method starts with the Caputo integral definition and then applies an approximation of the second derivative. The original method applies a three point central approximation, but we use a three point forward stencil. The reason for this was to avoid any problems near the boundary but the central approximation would in fact also work as throughout this thesis we always let derivatives on the boundary be equal to zero for $\alpha \in (1,2)$.

$${}_{C}D^{\alpha}_{a,x}u(x)\Big|_{x=x_n} = \frac{1}{\Gamma(2-\alpha)}\int_{x_0}^{x_n}(x_n-s)^{1-\alpha}u''(s)ds, \tag{2.49}$$

$$= \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}(x_n-s)^{1-\alpha}u''(s)ds, \tag{2.50}$$

$$\approx \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}\frac{u(x_k)-2u(x_{k+1})+u(x_{k+2})}{h^2}\int_{x_k}^{x_{k+1}}(x_n-s)^{1-\alpha}ds, \tag{2.51}$$

$$= \frac{h^{-2}}{\Gamma(3-\alpha)}\sum_{k=0}^{n-1}\left(u(x_k)-2u(x_{k+1})+u(x_{k+2})\right)\left((x_n-x_k)^{2-\alpha}-(x_n-x_{k+1})^{2-\alpha}\right), \tag{2.52}$$

$$= \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\sum_{k=0}^{n-1}\left(u(x_k)-2u(x_{k+1})+u(x_{k+2})\right)\left((n-k)^{2-\alpha}-(n-k-1)^{2-\alpha}\right). \tag{2.53}$$

Now, when we group terms by the functions $u(x_k)$, we get,

$$ {}_CD^{\alpha}_{a,x}u(x)\Big|_{x=x_n} \approx \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\sum_{k=0}^{n+1} u(x_k)W_L(k), \tag{2.54} $$

where the coefficients are given by,

$$ W_L(k) = \begin{cases} 1 & \text{if } k = n+1 \\ 2^{2-\alpha} - 3 & \text{if } k = n \\ (n-k+2)^{2-\alpha} - 3(n-k+1)^{2-\alpha} + 3(n-k)^{2-\alpha} - (n-k-1)^{2-\alpha} & \text{if } 2 \le k \le n-1 \\ -2(n^{2-\alpha} - (n-1)^{2-\alpha}) + (n-1)^{2-\alpha} - (n-2)^{2-\alpha} & \text{if } k = 1 \\ n^{2-\alpha} - (n-1)^{2-\alpha} & \text{if } k = 0. \end{cases} \tag{2.55} $$

We can approximate the right Caputo derivative in a similar way using a three point central stencil for the second derivative.

$$ {}_CD^{\alpha}_{x,b}u(x)\Big|_{x=x_n} = \frac{1}{\Gamma(2-\alpha)}\int_{x_n}^{x_{N+1}} (s-x_n)^{1-\alpha}u''(s)ds, \tag{2.56} $$

$$ = \frac{1}{\Gamma(2-\alpha)}\sum_{k=n}^{N}\int_{x_k}^{x_{k+1}} (s-x_n)^{1-\alpha}u''(s)ds, \tag{2.57} $$

$$ \approx \frac{1}{\Gamma(2-\alpha)}\sum_{k=n}^{N}\frac{u(x_{k+1}) - 2u(x_k) + u(x_{k-1})}{h^2}\int_{x_k}^{x_{k+1}} (s-x_n)^{1-\alpha}ds, \tag{2.58} $$

$$ = \frac{h^{-2}}{\Gamma(3-\alpha)}\sum_{k=n}^{N}\left(u(x_{k+1}) - 2u(x_k) + u(x_{k-1})\right)\left((x_{k+1}-x_n)^{2-\alpha} - (x_k - x_n)^{2-\alpha}\right), \tag{2.59} $$

$$ = \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\sum_{k=n}^{N}\left(u(x_{k+1}) - 2u(x_k) + u(x_{k-1})\right)\left((k+1-n)^{2-\alpha} - (k-n)^{2-\alpha}\right). \tag{2.60} $$

Then, when we group terms by the functions $u(x_k)$, we get,

$$ {}_CD^{\alpha}_{x,b}u(x)\Big|_{x=x_n} \approx \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\sum_{k=n-1}^{N+1} u(x_k)W_R(k), \tag{2.61} $$

where the coefficients are given by,

$$ W_R(k) = \begin{cases} 1 & \text{if } k = n-1 \\ 2^{2-\alpha} - 3 & \text{if } k = n \\ -(k-1-n)^{2-\alpha} - 3(k+1-n)^{2-\alpha} + 3(k-n)^{2-\alpha} + (k+2-n)^{2-\alpha} & \text{if } n+1 \le k \le N-1 \\ -(N-1-n)^{2-\alpha} - 2(N+1-n)^{2-\alpha} + 3(N-n)^{2-\alpha} & \text{if } k = N \\ (N+1-n)^{2-\alpha} - (N-n)^{2-\alpha} & \text{if } k = N+1. \end{cases} \tag{2.62} $$

Both the L1 and L2 methods originate from the Caputo definitions for fractional derivatives where we can use standard approximations for integer derivatives. This makes extending these methods a good option for working with non-uniform grids as we only need to adapt the approximation for the integer derivatives. This has already been done for the left Caputo derivative for $\alpha \in (0,1)$ [34] so in the following sections we will generalise the L1 and L2 methods to form the L1(NU) and L2(NU) methods to approximate the left and right Caputo derivatives on non-uniform grids.