which has the general solution

$$c_n(t) = c_n(0)\exp(-K_\alpha(\lambda_n)^{\frac{\alpha}{2}}t).$$

To obtain $c_n(0)$ we use the initial condition (10)

$$u(x,0) = \sum_{n=1}^{\infty} c_n(0)\sin\left(\frac{n\pi x}{L}\right) = g(x),$$

from which we deduce that

$$c_n(0) = \frac{2}{L}\int_0^L g(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi = b_n.$$

With this choice of the coefficients we have the analytic solution of the RFDE (8) based on Definition 2:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-K_\alpha\left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\alpha}{2}}t\right). \tag{11}$$

3.2. L2-approximation method for the RFDE

In this section, we provide a numerical solution of the RFDE (8) based on Definition 1 and Lemma 1 by the $L2$-approximation method. In Section 2, the Riesz fractional derivative of order $1 < \alpha \leqslant 2$ on a finite interval $[0,L]$ was defined in the sense of the Riemann–Liouville definition. There is another very important definition for the fractional derivative, namely, the Grünwald–Letnikov definition. For $1 < \alpha \leqslant 2$, the left- and right-handed Grünwald–Letnikov fractional derivatives on $[0,L]$ are, respectively, given by

$${}_0D_x^\alpha u(x) = \frac{u(0)x^{-\alpha}}{\Gamma(1-\alpha)} + \frac{u'(0)x^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}\int_0^x \frac{u^{(2)}(\xi)d\xi}{(x-\xi)^{\alpha-1}}, \tag{12}$$

$${}_xD_L^\alpha u(x) = \frac{u(L)(L-x)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{u'(L)(L-x)^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}\int_x^L \frac{u^{(2)}(\xi)d\xi}{(\xi-x)^{\alpha-1}}. \tag{13}$$

Following [3], there exists a link between these two approaches to differentiation of arbitrary real order. Let us suppose that the function $u(x)$ is $n-1$ times continuously differentiable in the interval $[0,L]$ and that $u^{(n)}(x)$ is integrable in $[0,L]$. Then for every $\alpha(0 \leqslant n-1 < \alpha < n)$ the Riemann–Liouville derivative exists and coincides with the Grünwald–Letnikov derivative. This relationship between the Riemann–Liouville and Grünwald–Letnikov definitions allows the use of the Riemann–Liouville definition during the problem formulation, and then the Grünwald–Letnikov definition for obtaining the numerical solution [14]. Therefore, using the link between these two definitions and the $L2$-algorithm proposed by [14,2], we can easily obtain the following numerical discretization scheme for the RFDE.

Assume that the spatial domain is $[0,L]$. The mesh is N equal intervals of $h = L/N$ and $x_l = lh$ for $0 \leqslant l \leqslant N$.

The second term of the right-hand side of Eq. (12) can be approximated by

$$\frac{u'(0)x^{1-\alpha}}{\Gamma(2-\alpha)} \approx \frac{h^{-\alpha}}{\Gamma(2-\alpha)l^{\alpha-1}}(u_1 - u_0).$$

The third term of the right-hand side of Eq. (12) can be approximated by

$$\begin{aligned}\frac{1}{\Gamma(2-\alpha)}\int_0^x \frac{u^{(2)}(\xi)d\xi}{(x-\xi)^{\alpha-1}} &= \frac{1}{\Gamma(2-\alpha)}\int_0^x \frac{u^{(2)}(x-\xi)d\xi}{\xi^{\alpha-1}} = \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{l-1}\int_{jh}^{(j+1)h}\frac{u^{(2)}(x-\xi)d\xi}{\xi^{\alpha-1}} \\ &\approx \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{l-1}\frac{u(x-(j-1)h) - 2u(x-jh) + u(x-(j+1)h)}{h^2}\int_{jh}^{(j+1)h}\frac{d\xi}{\xi^{\alpha-1}} \\ &= \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{l-1}(u_{l-j+1} - 2u_{l-j} + u_{l-j-1})\left[(j+1)^{2-\alpha} - j^{2-\alpha}\right],\end{aligned}$$

where $u(x-jh) = u_{l-j}$.

Hence, we obtain an approximation of the left-handed fractional derivative (12) with $1 < \alpha \leqslant 2$ as

$${}_0D_x^\alpha u(x_l) \approx \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\left\{\frac{(1-\alpha)(2-\alpha)u_0}{l^\alpha} + \frac{(2-\alpha)(u_1-u_0)}{l^{\alpha-1}} + \sum_{j=0}^{l-1}(u_{l-j+1} - 2u_{l-j} + u_{l-j-1})\left[(j+1)^{2-\alpha} - j^{2-\alpha}\right]\right\}. \tag{14}$$

Similarly, we can derive an approximation of the right-handed fractional derivative (13) with $1 < \alpha \leqslant 2$ as

$${}_xD_L^\alpha u(x_l) \approx \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\left\{\frac{(1-\alpha)(2-\alpha)u_N}{(N-l)^\alpha} + \frac{(2-\alpha)(u_N-u_{N-1})}{(N-l)^{\alpha-1}} + \sum_{j=0}^{N-l-1}(u_{l+j-1} - 2u_{l+j} + u_{l+j+1})\left[(j+1)^{2-\alpha} - j^{2-\alpha}\right]\right\}. \tag{15}$$