Shift matrices and Finite Differences

Cyclic shift matrix

upward shift: $S = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} \in \mathbb{R}^{N \times N}$

$S^T = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}$ (downward shift)

properties:

$\Rightarrow S^N = \underbrace{S S \cdots S}_{N \text{ times}} = I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

$\det(S) = (-1)^{N-1} \neq 0 \quad \begin{cases} \xRightarrow{\text{even } N} \det(S) = -1 \\ \xRightarrow{\text{odd } N} \det(S) = 1 \end{cases}$

S is an orthogonal matrix: $S S^T = I$

inverse of S: $S^{-1} = S^T = S^{N-1}$

eigenvalues of S: since $S^N = I \Rightarrow \lambda^N = 1$

$\Rightarrow \lambda_k = e^{\frac{2\pi i k}{N}} = \cos\left(\frac{2\pi k}{N}\right) + i \cdot \sin\left(\frac{2\pi k}{N}\right)$

$k = 0, 1, \dots, N-1$

characteristic polynomial: $p(\lambda) = \det(\lambda I - S^1)$

$= \lambda^N - 1$

eigenvectors: $\vec{v}_k = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ \lambda_k \\ \lambda_k^2 \\ \vdots \\ \lambda_k^{N-1} \end{pmatrix}$

class of circulant matrices: $\mathbb{C}$

each matrix $C \in \mathbb{C}$ can be written as:

$$C = c_0 I + c_1 S^1 + c_2 S^2 + \cdots + c_{N-1} S^{N-1}$$

for $\vec{x} \in \mathbb{R}^N$: $S^1 \vec{x} = S^1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_1 \end{pmatrix}$

it is a permutation matrix

and $S^{1T} \vec{x} = S^{-1} \vec{x} = \begin{pmatrix} x_N \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{pmatrix}$

S^k shifts k positions (upward)

and $(S^{1T})^k$ k positions downward

for $N > 1$: $\text{trace}(S^1) = 0$

since S^1 is orthogonal (unitary):

$$\begin{cases} \|S\|_2 = 1, \rho(S^1) = 1 \\ \text{matrix 2-norm} \quad \text{spectral radius} \\ \|S\vec{x}\|_p = \|\vec{x}\|_p \\ \text{vector p-norm} \end{cases}$$

Finite difference matrices

forward FD : $D_1^+ = \frac{1}{h}(S - I)$

backward FD : $D_1^- = \frac{1}{h}(I - S^T) = \frac{1}{h}(I - S^{-1})$

central FD : $D_1^C = \frac{1}{2h}(S - S^{-1}) = \frac{1}{2h}(S - S^T)$

$$D_2 = \frac{S + S^{-1} - 2I}{h^2} = \frac{S + S^T - 2I}{h^2}$$

general : $D_{1\gamma} = \frac{1}{2h}\left[(1-\gamma)S + 2\gamma I - (1+\gamma)S^T\right]$

$$= \frac{1}{2h}\begin{pmatrix} 2\gamma & 1-\gamma & & & -(1+\gamma) \\ -(1+\gamma) & 2\gamma & 1-\gamma & & \\ & -(1+\gamma) & 2\gamma & 1-\gamma & \\ & & \ddots & \ddots & \ddots \\ 1-\gamma & & & -(1+\gamma) & 2\gamma \end{pmatrix}$$

All FD-matrices commute :

example: $D_2 D_1^C = \frac{S + S^{-1} - 2I}{h^2} \cdot \frac{S - S^T}{2h}$

$$= \frac{1}{2h^3}\left[S^2 - SS^T + S^{-1}S - S^{-1}S^T - 2IS + 2IS^T\right]$$

$$= \frac{1}{2h^3}\left[S^2 - I + I - I - 2S + 2S^T\right] \quad (*)$$

and $D_1^C D_2 = \frac{S - S^T}{2h} \cdot \frac{S + S^{-1} - 2I}{h^2} = \frac{1}{2h^3}\left[S^2 + SS^{-1} - 2SI - S^TS - S^TS^{-1} + 2S^TI\right]$

$$= \frac{1}{2h^3}\left[S^2 + I - 2S - I - I + 2S^T\right]$$

$$= \frac{1}{2h^3}\left[S^2 - 2S - I + 2S^T\right] = (*)$$

In general this holds: $\begin{cases} D_2 D_1^c = P(S)\,Q(S) \\ D_1^c D_2 = Q(S)\,P(S) \end{cases}$

with P and Q polynomials in S.

"Laurent Algebra" $\mathcal{A} = \{P(S, S^{-1})\}$ Laurent polynomials in S

all FD-matrices $D_1^+, D_1^-, D_1^c, D_2, D_3, D_4, \ldots \in \mathcal{A}$

$\mathcal{A}$ is commutative: $AB = BA \quad \forall A, B \in \mathcal{A}$

Note: $D_2^3 \neq D_3^2$ $\left(\sim \frac{d^6}{dx^6}\right)$; as differential operators:

$$\left[\frac{d^2}{dx^2}\right]^3 = \frac{d^6}{dx^6} = \left[\frac{d^3}{dx^3}\right]^2$$

$$= \frac{d^2}{dx^2} \circ \frac{d^2}{dx^2} \circ \frac{d^2}{dx^2} \qquad = \frac{d^3}{dx^3} \circ \frac{d^3}{dx^3}$$

also: $(D_1^c)^4 \neq D_2^2$

$D_1^c D_3 \neq D_2^2$

$(D_1^+)^2 D_2 \neq D_2 (D_1^-)^2$

etcetera ...

formally: "$D_2 = \frac{d^2}{dx^2} + O(h^2)$" .....

"$D_3 = \frac{d^3}{dx^3} + O(h^2)$"

Laurent polynomial:

$$\underset{-M}{a} t^{-M} + \cdots + \underset{-n}{a} t^{-n} + \underset{-(n-1)}{a} t^{-(n-1)} + \cdots + \underset{-1}{a} t^{-1} + a_0 t + \cdots + a_n t^n + \cdots + \underset{K}{a} t^K$$

→ check the extra file on the webpage! and exercises 2