Lecture 3

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Numerical Methods for PDEs

Outline of Lecture 3

- exercises of Lecture 2
- method of horizontal/vertical lines
- time-integration methods
- local truncation error & consistency & zero stability
- absolute stability & stability regions
- boundary locus
- outlook to Lecture 4

Method of Lines [1]

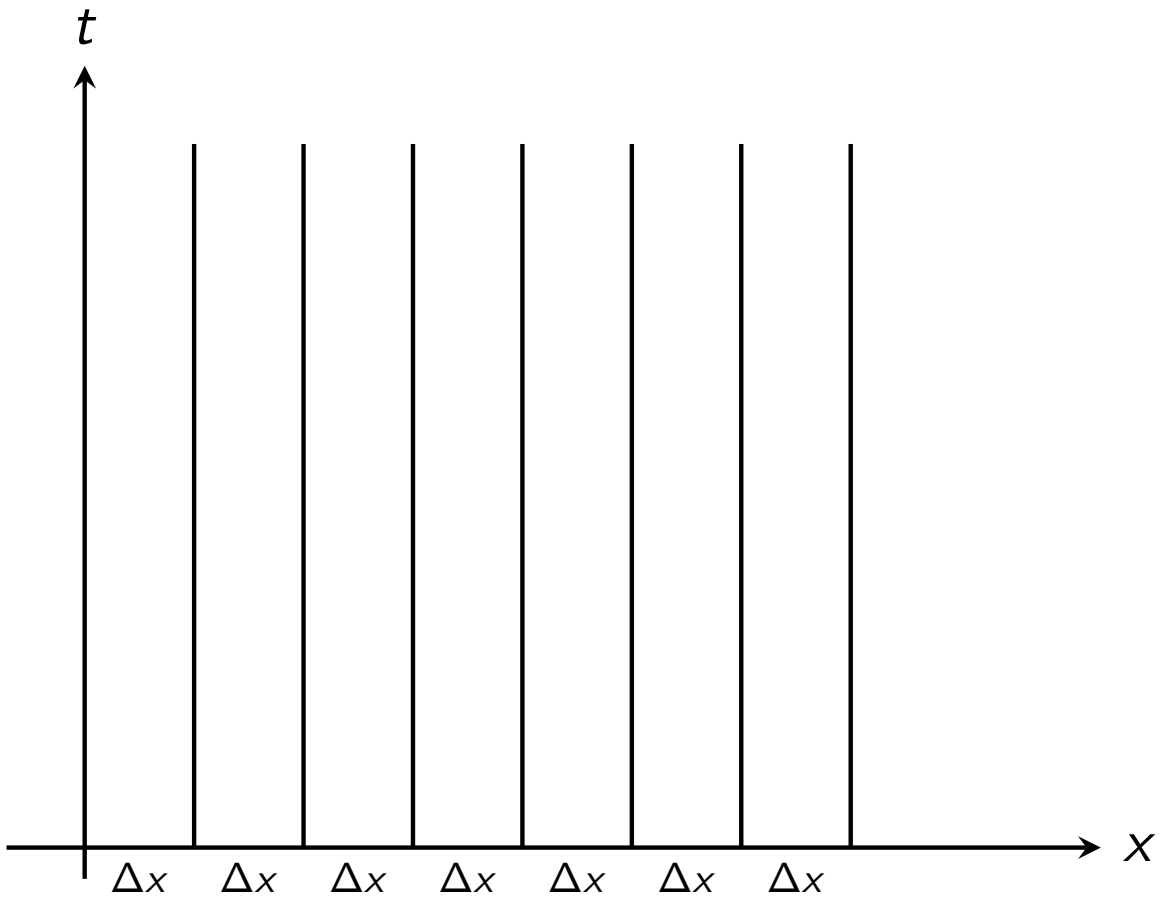
The spatial ("x") and temporal ("t") discretization are done separately in two steps.
Option 1: method of vertical lines ("the" method of lines)

1) spatial approximation of $\mathcal{L}(u) \implies \frac{d}{dt}\vec{u}(t) = \vec{L}(\vec{u}(t))$

$$\begin{cases} \text{a system of} \\ \text{ordinary differential equations} \\ \text{(ODEs)} \end{cases}$$

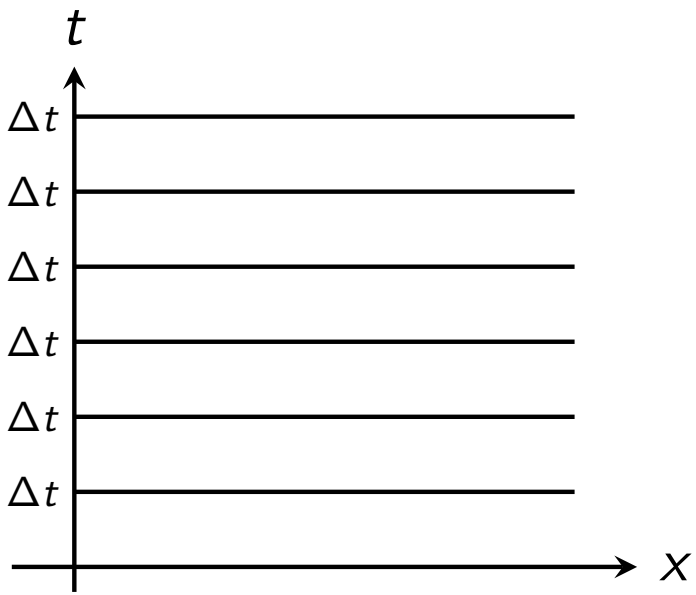
2) time integration
$\Downarrow$
fully discrete approximation

Method of Lines [2]



Method of Lines [3]

Option 2: method of horizontal lines ("Rothe's method")



1) time integration
$\Longrightarrow \vec{F}(\vec{u}^n, \vec{u}^{n+1}) = \vec{0}$

$\begin{cases} \text{a system of (non)linear} \\ \text{"stationary" PDEs} \end{cases}$

2) numerical (non)linear algebra ↘
$\longrightarrow$ fully discrete approximation

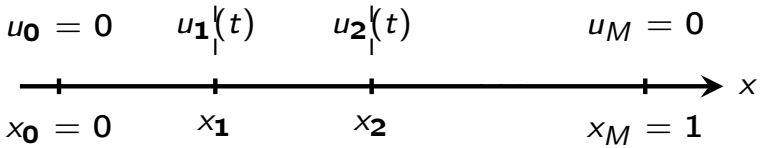
Method of Lines [4]

We will use **option 1 "MoL"**

example 1 the heat equation $u_t = \kappa \cdot u_{xx} \quad , \quad x \in (0,1), \quad t > 0, \kappa > 0$

IC: $u(x,0) = \sin(\pi x), \quad x \in [0,1]$ $\qquad u(x,t) = ?$
BCs: $u(0,t) = u(1,t) = 0, \quad t > 0$ (homogeneous Dirichlet))

step 1 in MoL : $u_i(t) \approx u(x_i, t)$



Method of lines [5]

$$x_i = \frac{i}{M}, \quad i = 0, \ldots, M \quad \longleftarrow \text{number of spatial grid points}$$

$$\Longrightarrow \; x_{i+1} - x_i = \frac{i+1}{M} - \frac{i}{M} = \frac{1}{M} \quad \forall i \quad \underset{\text{def}}{=} \Delta x \text{ (constant)}$$

BCs: $u_0 = 0 \quad \forall t$

$u_M = 0 \quad \forall t$

IC: $u_i(0) = \sin(\pi x_i), i = 1, \ldots, M-1$

Method of lines [6]

How to obtain ODE equations for $u_i(t),\ t > 0$?

$\rightsquigarrow$ approximate $u_{xx}(x_i, t)$ in terms of the $u_i(t)$:

$$u_{xx}(x_i, t) \approx \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{(\Delta x)^2}, \qquad \begin{matrix} i = 1, \ldots, M-1 \\ t > 0 \end{matrix}$$

$\Longrightarrow$ system of $M-1$ **coupled ODEs:** $\dot{\vec{u}}(t) = \kappa \cdot D_{2c}\vec{u}(t)$

with $\vec{u}(t) = (u_1(t), u_2(t), \ldots, u_{M-1}(t))^T$ and

$$D_{2c} = \frac{1}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{pmatrix}$$

a tri-diagonal matrix with $\lambda \in \mathbb{R}^-$

Method of lines [7]

step 2 in MoL : numerically integrate the ODE system

example: use Euler Forward (EF) : $\frac{\vec{u}^{n+1}-\vec{u}^n}{\Delta t} = \kappa \cdot D_{2c}\vec{u}^n$

$$n = 0, 1, 2, \ldots, N-1$$

with $u_i^N \approx u(x_i, T)$

$\uparrow$

final time

$$\Longrightarrow \begin{cases} \vec{u}^{n+1} = (\mathcal{I} + \kappa \Delta t D_{2c})\vec{u}^n \\ \vec{u}^0 \text{ given by IC} \end{cases}$$

$$\Delta t = \frac{T}{N}$$

$$t^n = n \cdot \Delta t, \;\; n = 0, 1, \ldots, N$$

$$u_i^n \approx u(x_i, t^n)$$

Method of lines [8]

or **Euler Backward (EB):** $\frac{\vec{u}^{n+1}-\vec{u}^{n}}{\Delta t} = \kappa \cdot D_{2c}\vec{u}^{n+1}$

$$\Longrightarrow \begin{cases} \vec{u}^{n+1} = (\mathcal{I} - \kappa \Delta t D_{2c})^{-1}\vec{u}^{n} \\ \vec{u}^{0} \text{ given by IC} \end{cases}$$

or .....many options.....

Method of lines [9]

In general: suppose $\mathcal{L}$ is a linear spatial operator

$$u_t = \mathcal{L}(u)$$

Semi-discretize in space:

$\Longrightarrow$ $\dot{\vec{u}}(t) = L_{\Delta x}\vec{u}(t)$ with finite-difference matrix $L_{\Delta x}$

Approximate in time (EF, EB, ...)

$\downarrow$ or: "exact" semi-discrete solution

$$\vec{u}(t) = e^{t\cdot L_{\Delta x}}\vec{u}_0$$

$\sim$ matrix exponential

"exponential integrators" ..... $\rightsquigarrow$ .....?

Method of lines [10]

example 2 the advection equation

$$\begin{cases} u_t + a u_x = 0 & , \ x \in (0,1), \ t > 0 \\ u(x,0) = u_0(x) & , \ x \in (0,1), \ t = 0 \end{cases}$$

(exact solution: $u(x,t) = u_0(x - at)$)

If $a > 0$, then we need a BC at $x = 0$ ("the inflow boundary") in that case $x = 1$ is the "outflow boundary".

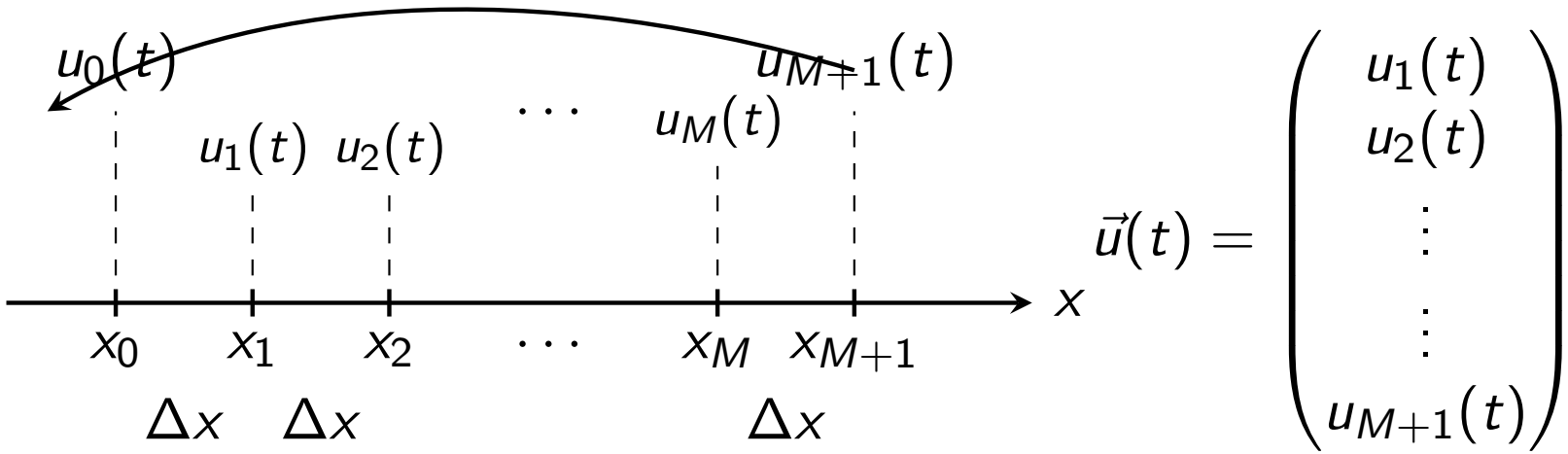
If $a < 0$; BC at $x = 1$

consider periodic BCs : $u(0,t) = u(1,t) \quad , \quad t > 0$
("whatever flows at the outflow boundary, flows back in at the inflow boundary")

Method of lines [11]

In that case: $u_0(t) = u_{M+1}(t)$

$$\downarrow$$



$$\vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ \vdots \\ u_{M+1}(t) \end{pmatrix}$$

Method of lines [12]

$$u_x(x_i,t) \approx \frac{u_{i+1}(t)-u_{i-1}(t)}{2\Delta x}, \quad i=2,3,\ldots,M$$

$\uparrow$

Central FDs (example)

$\Longrightarrow$ **ODE system:** (semi-discrete)

$$\dot{u}_i(t) = -\frac{a}{2\Delta x}\left(u_{i+1}(t)-u_{i-1}(t)\right), \quad i=2,\ldots,M$$

with $\dot{u}_1(t) = -\frac{a}{2\Delta x}\left(u_2(t)-u_0(t)\right)$

$\downarrow$

$(u_0 = u_{M+1})$ $\quad = -\frac{a}{2\Delta x}\left(u_2(t)-u_{M+1}(t)\right)$

and $\dot{u}_{M+1}(t) = -\frac{a}{2\Delta x}\left(u_{M+2}(t)-u_M(t)\right)$

$\downarrow$

$(u_1 = u_{M+2})$ $\quad = -\frac{a}{2\Delta x}\left(u_1(t)-u_M(t)\right)$

Method of lines [13]

$$\Rightarrow \dot{\vec{u}}(t) = a \cdot D_{1c}\vec{u}(t)$$

with $D_{1c} = \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{pmatrix}$

a circulant, skew-symmetric matrix

$\left(D_{1c}^T = -D_{1c}\right)$

$$\Longrightarrow \lambda \in i\mathbb{R}$$

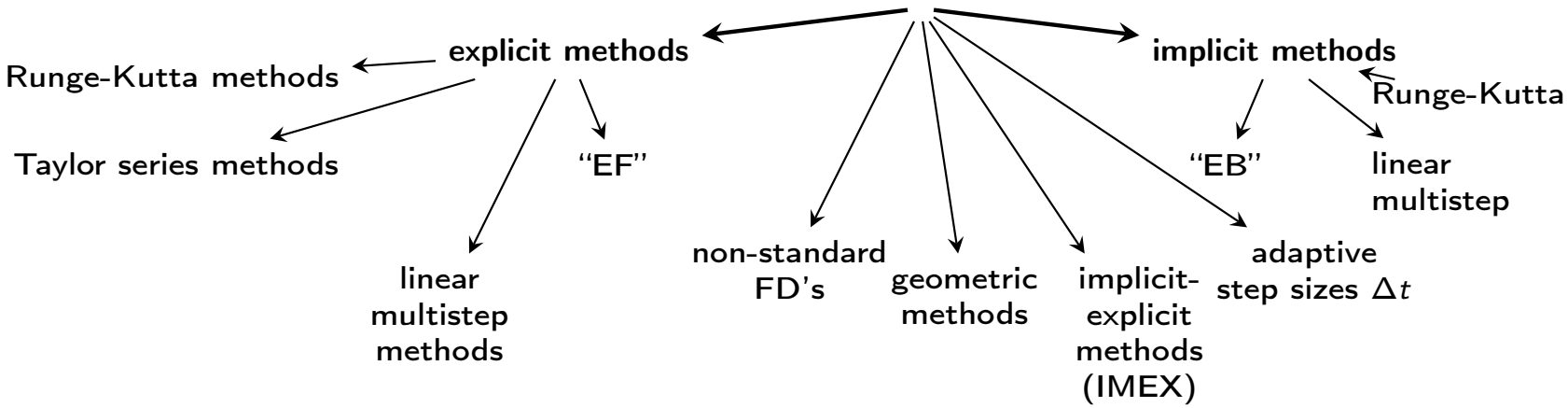
Method of lines [14]

In general, we obtain, after one step in MoL:

$$\begin{cases} \dot{\vec{u}}(t) = M\vec{u}(t) \\ \vec{u}(0) = \vec{u}_0 \end{cases} \longleftarrow$$

the eigenvalues of the matrix M are going to play an important role in step 2 of MoL.

In step 2, we need to apply a time-integration method:



- accuracy ?
- stability ?
- efficiency ?

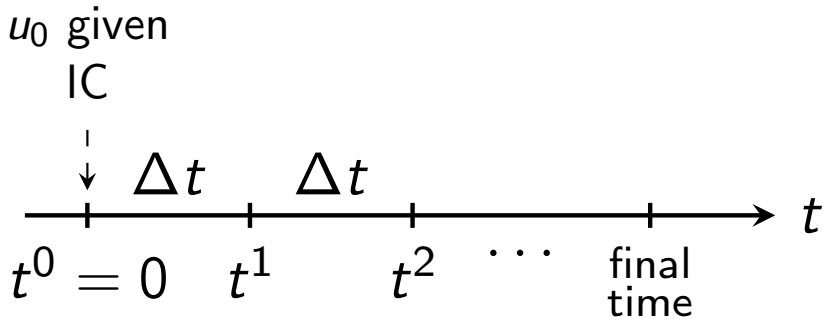
$\Longleftarrow$

!

Time-integration methods [1]

Consider the scalar ODE : $u' = f(u)$

(autonomous)



$$t^n = n \cdot \Delta t, \; n \geq 0$$
$$u^n \approx u(t^n)$$

Time-integration methods [2]

Forward Euler :
(EF, Explicit Euler)

$$u^{n+1} = u^n + \Delta t\, f(u^n)$$

Backward Euler :
(EB, Implicit Euler)

$$u^{n+1} = u^n + \Delta t\, f(u^{n+1})$$

Trapezoidal method :
(implicit)

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}\left[f(u^n) + f(u^{n+1})\right]$$

Midpoint method :
("leapfrog")
explicit

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = f(u^n)$$

$\mathbf{BDF_2}$:
(implicit)

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} = f(u^{n+1}) \quad \ldots\ldots \text{etcetera} \ldots\ldots$$

Time-integration methods [3]

Truncation error : (example : midpoint, leapfrog)

$$\text{TE}: \quad \tau^n \overset{\text{def}}{=} \frac{u(t^{n+1}) - u(t^{n-1})}{2\Delta t} - f(u(t^n))$$

$$\xrightarrow{\text{Taylor expansion}} \quad = \left[u'(t^n) + \tfrac{1}{6}(\Delta t)^2 u'''(t^n) + \mathcal{O}((\Delta t)^4)\right] - u'(t^n)$$

$$= \tfrac{1}{6}(\Delta t)^2 u'''(t^n) + \mathcal{O}((\Delta t)^4)$$

(note that the $\mathcal{O}((\Delta t)^3)$ term drops out by "symmetry")

[similar for other methods]

Taylor series methods :

$$u(t^{n+1}) \approx u(t^n) + \Delta t\, u'(t^n) + \frac{1}{2}(\Delta t)^2 u''(t^n) + \cdots + \frac{1}{p!}(\Delta t)^p u^{(p)}(t^n)$$

we know : $u' = f(u) \implies u'' = \dfrac{df}{du} \cdot u' = \dfrac{df}{du} \cdot f(u)$ etcetera for $u''', \ldots$

Time-integration methods [4]

two-stage (explicit)

$$\begin{cases} k_{\mathbf{1}} = u^n + \frac{\mathbf{1}}{\mathbf{2}} \Delta t\, f(u^n) \\ u^{n+\mathbf{1}} = u^n + \Delta t\, f(k_{\mathbf{1}}) \end{cases} \longleftarrow$$

intermediate value

to approximate $u(t^{n+\frac{\mathbf{1}}{\mathbf{2}}})$

using EF

$$\Longrightarrow$$

local error: $\tau^n = \mathcal{O}((\Delta t)^{\mathbf{3}})$ ("second order"); TE vs LTE...

global error: $\mathcal{O}((\Delta t)^{\mathbf{2}})$

four-stage (explicit)

$$\begin{cases} k_{\mathbf{1}} = u^n \\ k_{\mathbf{2}} = u^n + \frac{\mathbf{1}}{\mathbf{2}} \Delta t\, f(k_{\mathbf{1}}) \\ k_{\mathbf{3}} = u^n + \frac{\mathbf{1}}{\mathbf{2}} \Delta t\, f(k_{\mathbf{2}}) \\ k_{\mathbf{4}} = u^n + \Delta t\, f(k_{\mathbf{3}}) \\ u^{n+\mathbf{1}} = u^n + \frac{\Delta t}{\mathbf{6}} \left[f(k_{\mathbf{1}}) + 2f(k_{\mathbf{2}}) + 2f(k_{\mathbf{3}}) + f(k_{\mathbf{4}}) \right] \end{cases}$$

"fourth-order" method: global error: $\mathcal{O}((\Delta t)^{\mathbf{4}})$ (local error: $\mathcal{O}((\Delta t)^{\mathbf{5}})$)

$\longrightarrow$ can be generalized

- r-stage
- implicit
- many variants

Time-integration methods [5]

Linear Multistep Methods (LMM)

r-step LMM: $\sum_{j=0}^{r} \alpha_j u^{n+j} = \Delta t \sum_{j=0}^{r} \beta_j f(u^{n+j})$

if $\beta_r = 0$, then explicit (otherwise: implicit)

Example Adams methods :

$$u^{n+r} = u^{n+r-1} + \Delta t \sum_{j=0}^{r} \beta_j f(u^{n+j})$$

$$(\alpha_r = 1,\ \alpha_{r-1} = -1,\ \alpha_j = 0 \text{ for } j < r-1)$$

3-step explicit:

$$u^{n+3} = u^{n+2} + \frac{\Delta t}{12}\left[5\, f(u^n) - 16\, f(u^{n+1}) + 23\, f(u^{n+2})\right]$$

2-step implicit:

$$u^{n+2} = u^{n+1} + \frac{\Delta t}{2}\left[-f(u^n) + 8\, f(u^{n+1}) + 5\, f(u^{n+2})\right]$$

Time-integration methods [6]

$$\text{LTE} \quad \tau^{n+r} = \frac{1}{\Delta t}\left(\sum_{j=0}^{r} \alpha_j u(t^{n+j}) - \Delta t \sum_{j=0}^{r} \beta_j \underbrace{u'(t^{n+j})}_{=f(u^{n+j})} \right)$$

consistency $\quad \tau \longrightarrow 0 \quad \text{as } \Delta t \longrightarrow 0$

$$\iff \quad \sum_{j=0}^{r} \alpha_j = 0 \quad \& \quad \sum_{j=0}^{r} j\,\alpha_j = \sum_{j=0}^{r} \beta_j$$

Time-integration methods [7]

Characteristic polynomials

$$\rho(\zeta) = \underbrace{\sum_{j=0}^{r} \alpha_j \zeta^j}_{\text{degree}=r} \quad \& \quad \sigma(\zeta) = \underbrace{\sum_{j=0}^{r} \beta_j \zeta^j}_{\substack{\text{degree}=r \text{ (if implicit),} \\ \text{otherwise degree}<r}}$$

note: $\rho(1) = \sum_{j=0}^{r} \alpha_j \quad \& \quad \rho'(\zeta) = \sum_{j=1}^{r} j\, \alpha_j \zeta^{j-1}$

consistency $\iff \rho(1) = 0 \quad \text{AND} \quad \rho'(1) = \sigma(1)$

Zero stability [1]

Example of a consistent LMM that does not converge

$$u^{n+2} - 3u^{n+1} + 2u^n = -\Delta t\, f(u^n) \qquad [*]$$

LTE $$\tau^n = \frac{1}{\Delta t}\left[u(t^{n+2}) - 3u(t^{n+1}) + 2u(t^n)\right] + u'(t^n)$$

$$= \frac{5}{2}\Delta t\, u''(t^n) + \mathcal{O}((\Delta t)^2) \quad \longrightarrow 0 \text{ for } \Delta t \to 0$$

$$\Longrightarrow \text{ method is consistent and first order accurate}$$

What happens with the global error?

Zero stability [2]

Check with the "trivial" ODE: $\begin{cases} u'(t) = 0 \\ u(0) = 0 \end{cases} \implies u(t) = 0 \quad \forall t \geq 0$!

Apply LMM to equation $[*]$: $u^{n+2} - 3u^{n+1} + 2u^n = 0 \qquad [**]$

we need two starting values, say $u^0 = u^1 = 0 \implies u^n = 0 \quad \forall n \geq 0$ (solves the "ODE" exactly)

However, in general, some perturbation must be added,

say $u^0 = 0$, $u^1 = \underbrace{\Delta t}_{\textbf{small!!}} \longrightarrow u^{20} = \mathcal{O}(10^4)$ **blows-up**

Explanation: solve $[**]$ explicitly $\implies u^n = 2u^0 - u^1 + 2^n(u^1 - u^0)$ (check!!)

we know that $u(t) = 0, \forall t \geq 0 \implies$ global error $= u^n - \underbrace{u(t^n)}_{=0} = u^n$

Zero stability [3]

factor 2^n magnifies the global error $\Longrightarrow$ no convergence

for $n \to \infty$!!

"Root condition": the roots of $\rho(\zeta)$ are all inside the unit circle in $\mathbb{C}$

(those on the circle: simple roots)

A method is zero-stable $\Longleftrightarrow$ it satisfies the root condition

Theorem: method is convergent $\Longleftrightarrow$ it is consistent **AND** zero-stable

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Note: zero-stability $\Longleftrightarrow$ the origin $z = 0 \in \mathbb{C}$ lies in the (absolute) stability region

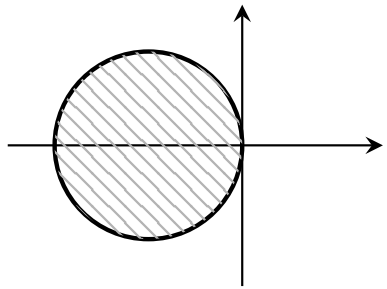
Stability regions [1]

Define the stability polynomial $\pi(\zeta; z) = \rho(\zeta) - z \cdot \sigma(\zeta)$

The region of (absolute) stability ("stability region") $\Longleftrightarrow$ "A-stability" of a method
$= \{z \in \mathbb{C} \mid \pi(\zeta; z)$ satisfies the "root condition"$\}$

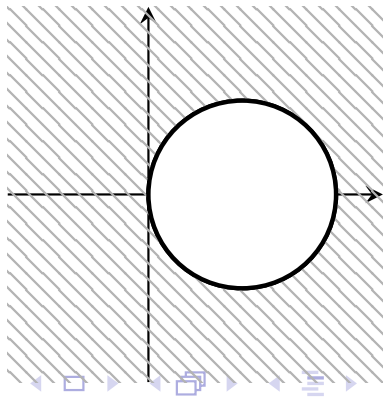
EF: $\pi = \zeta - (1 + z)$, a single root: $\zeta_1 = 1 + z$

$|\zeta_1| = |1 + z| < 1$
inside a circle



EB: $\pi = (1 - z)\zeta - 1$, a single root: $\zeta_1 = (1 - z)^{-1}$

$|\zeta_1| = |(1 - z)^{-1}| < 1$
outside another circle



Stability regions [2]

Absolute stability $\sim$ **stability properties of the ODE methods** in step 2 of MoL and the properties of the semi-discrete matrix M

A method is (absolutely) stable, or A-stable,
if $\lambda \Delta t \stackrel{\text{def}}{:=} z (\in \mathbb{C})$ lies in $S \longleftarrow$ **stability region** of the ODE method, where $\lambda \in$ spectrum of the matrix M

Stability regions [3]

Consider the scalar "test" ODE : $\begin{cases} u' = \lambda u \\ u(0) = u_0 \end{cases}$ with $\lambda \in \mathbb{C}$

and also the "perturbed" ODE: $\begin{cases} v' = \lambda v \\ v(0) = u_0 + \delta \end{cases}$

with small perturbation δ

As an example, apply EF:

$$\begin{cases} u^{n+1} = (1 + \lambda \Delta t) u^n \\ \qquad = \cdots = (1 + \lambda \Delta t)^{n+1} u^0 \\ v^{n+1} = (1 + \lambda \Delta t) v^n \\ \qquad = \cdots = (1 + \lambda \Delta t)^{n+1} (u^0 + \delta) \end{cases}$$

Stability regions [4]

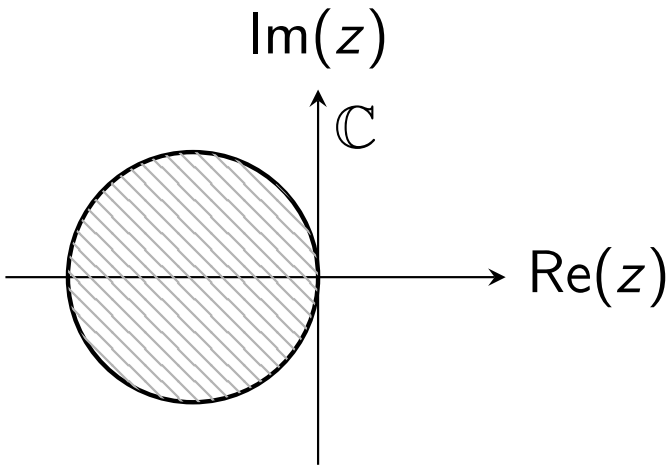
$$\Rightarrow \underbrace{v^{n+1} - u^{n+1}}_{\text{? small ??}} = \underbrace{(1+\lambda\Delta t)^{n+1}}_{\substack{\text{amplification factor} \\ |\cdot|>1 \text{ or } |\cdot|<1 \text{ ?}}} \cdot \underbrace{\delta}_{\text{small}}$$

If $|1+\lambda\Delta t| = |1+z| < 1$, then $|v^{n+1} - u^{n+1}| < 1$

and the initial small perturbation won't affect the numerical solution.

Stability regions [5]

EF is stable, if $|1+z|<1$ with $z = \underbrace{\lambda}_{\text{from ODE}} \star \underbrace{\Delta t}_{\text{from numerical method}}$



$$S_{\text{EF}} = \{z \in \mathbb{C} \mid |1+z| < 1\}$$

$R(z) = 1+z \approx 1+z+\frac{1}{2}z^2+\cdots = e^z$ is called the "stability function" of the method

Note: solutions of the ODE are "stable" for $\lambda \in \mathbb{C}^-$

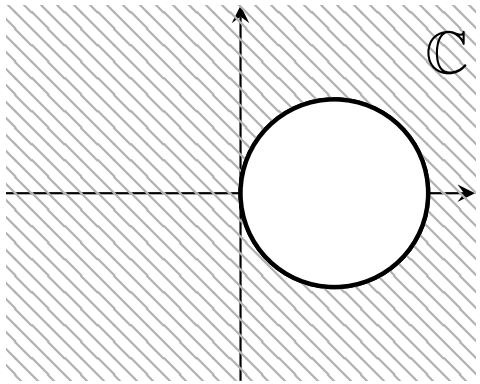
$\Rightarrow$ EF is conditionally stable with accuracy $\mathcal{O}(\Delta t)$

Stability regions [6]

For Euler-Backward (EB): $S_{\text{EB}} = \{z \in \mathbb{C} \mid |1 - z| > 1\}$

$$R(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \approx 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots = e^z$$

Since $S_{\text{EB}} \supset \mathbb{C}^-$, EB is unconditionally stable.
Accuracy: $\mathcal{O}(\Delta t)$

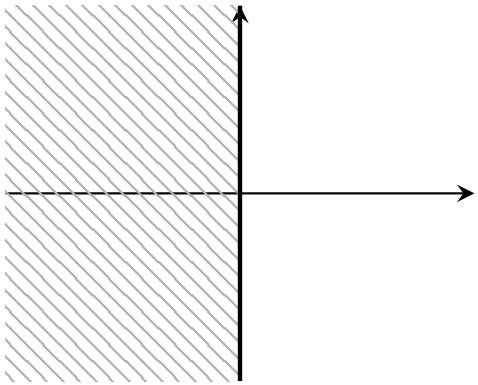


Stability regions [7]

Trapezoïdal method: $S_{\mathrm{TM}} = \mathbb{C}^-$

unconditionally stable
accuracy: $\mathcal{O}((\Delta t)^2)$

$$R(z) = \frac{1+\frac{1}{2}z}{1-\frac{1}{2}z} = (1+\tfrac{1}{2}z)(1+\tfrac{1}{2}z+\dots) = 1+z+\cdots \approx e^z$$



Stability regions [8]

p-th order Taylor method :

$$R(z) = 1 + z + \frac{1}{2!}z^2 + \cdots + \frac{1}{p!}z^p \approx e^z$$

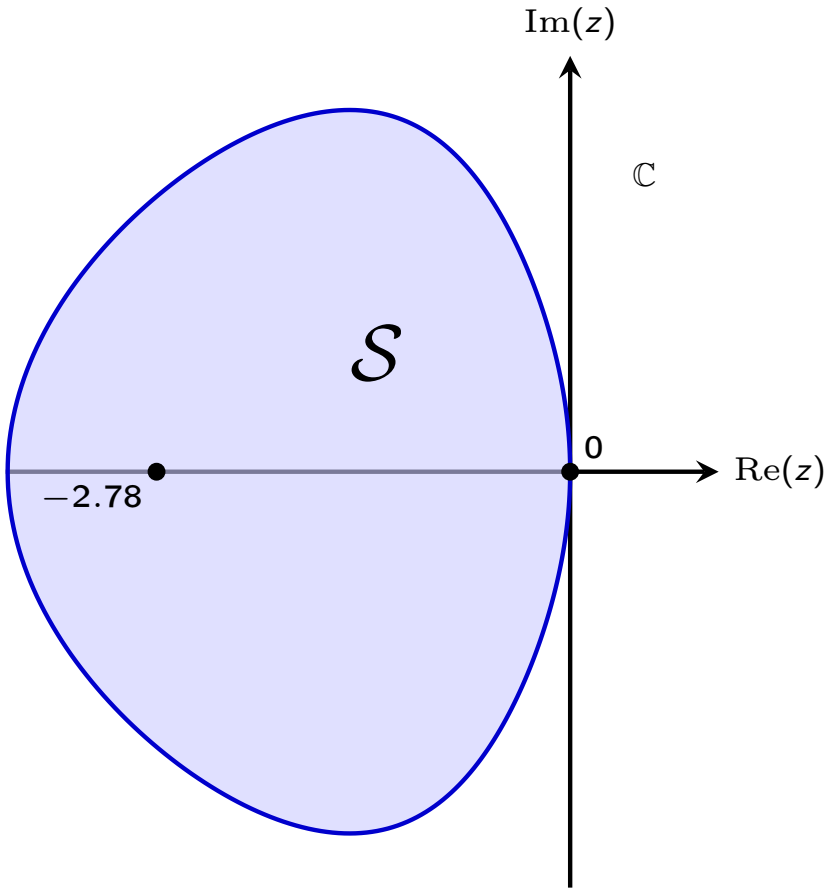
- conditionally stable
- accuracy : $\mathcal{O}((\Delta t)^p)$
- explicit

Stability regions [9]

Runge-Kutta 4 (RK4) :

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 \approx e^z$$

conditionally stable
accuracy : $\mathcal{O}((\Delta t)^4)$
explicit



Stability regions [10]

Note: $R(z)$ is a rational function or polynomial in z Theorem :

The boundary locus (= curve in $\mathbb{C}$ defined by $|R(z)| = 1$) is symmetric around the real axis in the complex plane:

$$R(\bar{z}) = \overline{R(z)}$$

and

the angle at $z = 0 \in \mathbb{C}$ between the boundary locus and the real axis in the complex plane is $90°$:

$$\arg(R'(0)) = \tfrac{\pi}{2}$$

Boundary locus [1]

how to find the region of (absolute) stability ?

$z \in$ stability region S, if the stability polynomial $\pi(\zeta; z)$ satisfies the "root condition" for this $z \in \mathbb{C}$.

If $z \in \partial S$, then $\pi(\zeta; z)$ must have at least one root ζ_j with $|\zeta_j| = 1$.
this ζ_j must have the form: $\zeta_j = e^{i\varphi}$ for some $\varphi \in [0, 2\pi]$.

$$
\begin{aligned}
&\Longrightarrow \quad \pi(e^{i\varphi}; z) = 0 \text{ for this } \varphi, z \text{ combination} \\
&\Longleftrightarrow \quad \rho(e^{i\varphi}) - z\,\sigma(e^{i\varphi}) = 0 \\
&\Longleftrightarrow \quad z = \frac{\rho(e^{i\varphi})}{\sigma(e^{i\varphi})} \qquad \text{(from } \varphi \text{ follows } z \in \partial S)
\end{aligned}
$$

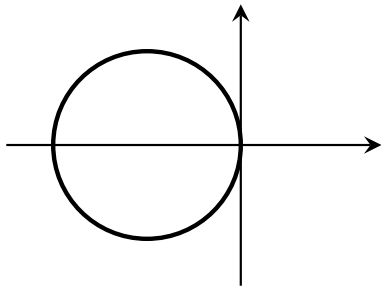
Boundary locus [2]

Each point on ∂S must be of this form ! (for some value of $\varphi \in [0, 2\pi]$)

$$\Longrightarrow \text{ "simply" plot the parametrized curve } z = \frac{\rho(e^{i\varphi})}{\sigma(e^{i\varphi})}, \ \varphi \in [0, 2\pi]$$

example 1: EF $\begin{cases} \rho(\zeta) = \zeta - 1 \\ \sigma(\zeta) = 1 \end{cases} \Longrightarrow z(\varphi) = \frac{e^{i\varphi} - 1}{1} = e^{i\varphi} - 1$

this function maps $[0, 2\pi]$ to the unit circle centered at $z = -1$.



$\star$ Evaluate at some "random" point z on the inside or

outside of the curve and see if the polynomial satisfies the "root condition"

Boundary locus [3]

example 2 $\quad$ BDF$_2$ method $\quad u^{n+2} - \frac{4}{3}u^{n+1} + \frac{1}{3}u^n = \frac{2}{3}\Delta t\, f(u^{n+2})$

$$\begin{cases} \rho(\zeta) = \zeta^2 - \frac{4}{3}\zeta + \frac{1}{3} \\ \sigma(\zeta) = \frac{2}{3}\zeta^2 \end{cases} \quad \Longrightarrow \quad \pi(\zeta; z) = \left(1 - \tfrac{2}{3}z\right)\zeta^2 - \tfrac{4}{3}\zeta + \tfrac{1}{3}$$

$$\Longrightarrow \quad z = \frac{e^{2i\varphi} - \frac{4}{3}e^{i\varphi} + \frac{1}{3}}{\frac{2}{3}e^{2i\varphi}} = \frac{3}{2} - 2e^{-i\varphi} + \frac{1}{2}e^{-2i\varphi}, \quad z \in \partial S$$

set $z = x + iy$, $\quad \begin{cases} x = x(\varphi) \\ y = y(\varphi) \end{cases} \quad \Longrightarrow \quad x + iy = \frac{3}{2} - 2\left(\cos(-\varphi) + i\sin(-\varphi)\right)$

$$+\frac{1}{2}\left(\cos(-2\varphi) + i\sin(-2\varphi)\right)$$

Boundary locus [4]

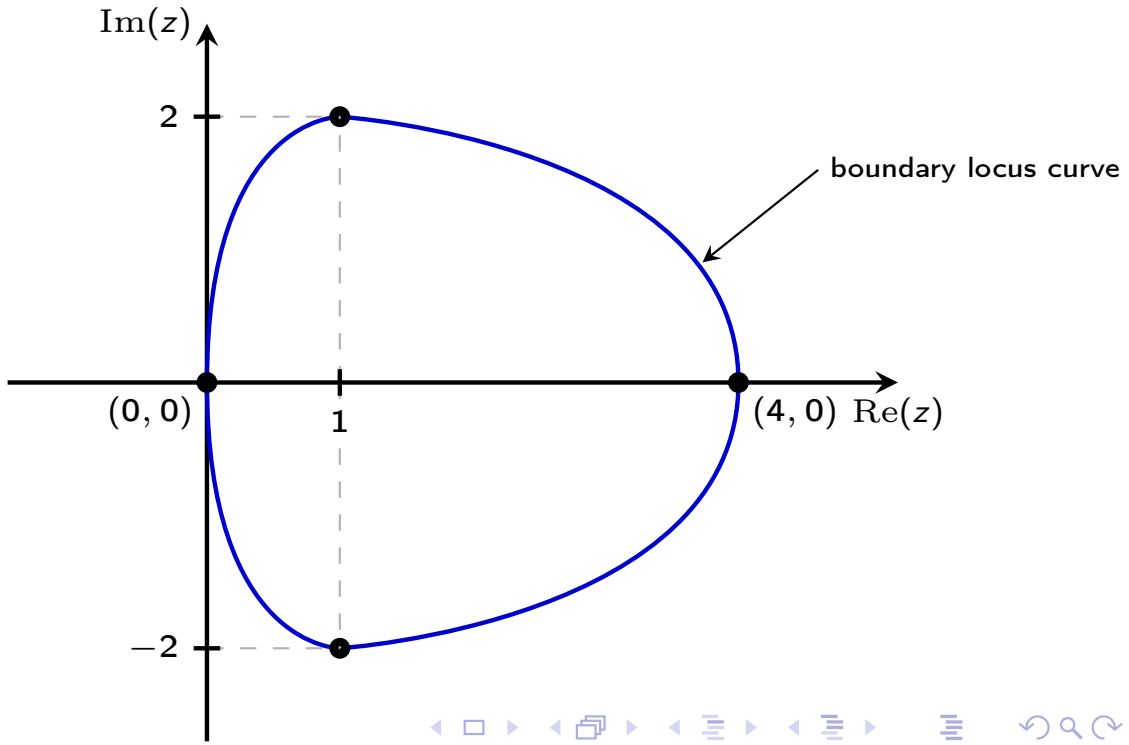
$$\Longrightarrow \begin{cases} x(\varphi) = \frac{3}{2} - 2\cos(\varphi) + \frac{1}{2}\cos(2\varphi) \\ y(\varphi) = 2\sin(\varphi) - \frac{1}{2}\sin(2\varphi) \end{cases}$$

$$= \frac{3}{2} - 2\cos(\varphi) + \frac{1}{2}\cos(2\varphi) + i\left(2\sin(\varphi) - \frac{1}{2}\sin(2\varphi)\right)$$

this is a curve in the x-y-plane ("$\mathbb{C}$")

Table:

φ	x	y
0	0	0
$\pi/2$	1	2
π	4	0
$3\pi/2$	1	-2
2π	0	0



Boundary locus [5]

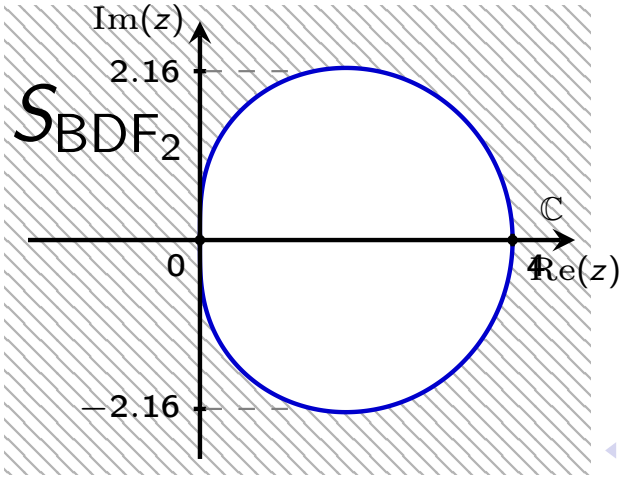
Stability region S : inside or outside closed curve ? (need to check one point)

Recall $\pi(\zeta; z) = \left(1 - \frac{2}{3}z\right)\zeta^2 - \frac{4}{3}\zeta + \frac{1}{3} \implies \pi\left(\zeta; -\frac{3}{2}\right) = 2\zeta^2 - \frac{4}{3}\zeta + \frac{1}{3}$

has roots : $\zeta_{1,2} = \dfrac{\frac{4}{3} \pm \sqrt{\left(-\frac{4}{3}\right)^2 - 4 \cdot 2 \cdot \frac{1}{3}}}{2 \cdot 2} = \frac{1}{3} \pm \frac{i}{4}\frac{\sqrt{8}}{3}$

and $|\zeta|^2 = \frac{1}{9} + \frac{8}{16 \cdot 9} = \frac{1}{9} + \frac{1}{18} = \frac{3}{18} \implies |\zeta| < 1$

$\implies$ S outside ∂S



Outlook to Lecture 4

- prepare exercises of Lecture 3 (see webpage!)
- the heat equation
- semi-discretization
- time-integration
- space-time discretizations
- higher dimensions