Lecture 4

Paul Andries Zegeling

Department of Mathematics, Utrecht University

Numerical Methods for PDEs

Outline of Lecture 4

- exercises of Lecture 3
- FTCS for the heat equation
- Von Neumann stability
- Lax equivalence theorem
- Conditional consistency and unconditional instability
- The heat equation in 2D
- outlook to Lecture 5

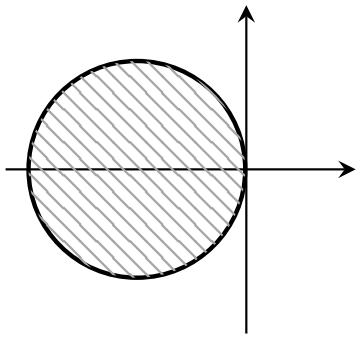
FTCS for heat equation [1]

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}$$

↖ diffusion coefficient > 0

(1) **Method of Lines** $\underset{\text{step 1}}{\Longrightarrow}$ $\frac{d\vec{u}}{dt} = k \cdot D_{2c}\vec{u}$; $\lambda(D_{2c}) \in \mathbb{R}^-$ (see exercise)

step 2 (for example) EF stability region in $\mathbb{C}$: S_{EF}



check . . .

$\Longrightarrow$ $k \cdot \lambda(D_{2c}) \in \underset{\text{(stable num. sol.)}}{S_{\text{EF}}}$ if $\boxed{\frac{k \cdot \Delta t}{(\Delta x)^2} \leq \frac{1}{2}}$ conditional stability

FTCS for heat equation [2]

2) **space-time discretization** ("one" step) $\Longleftrightarrow$ MoL + EF (for this choice)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = k \cdot \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

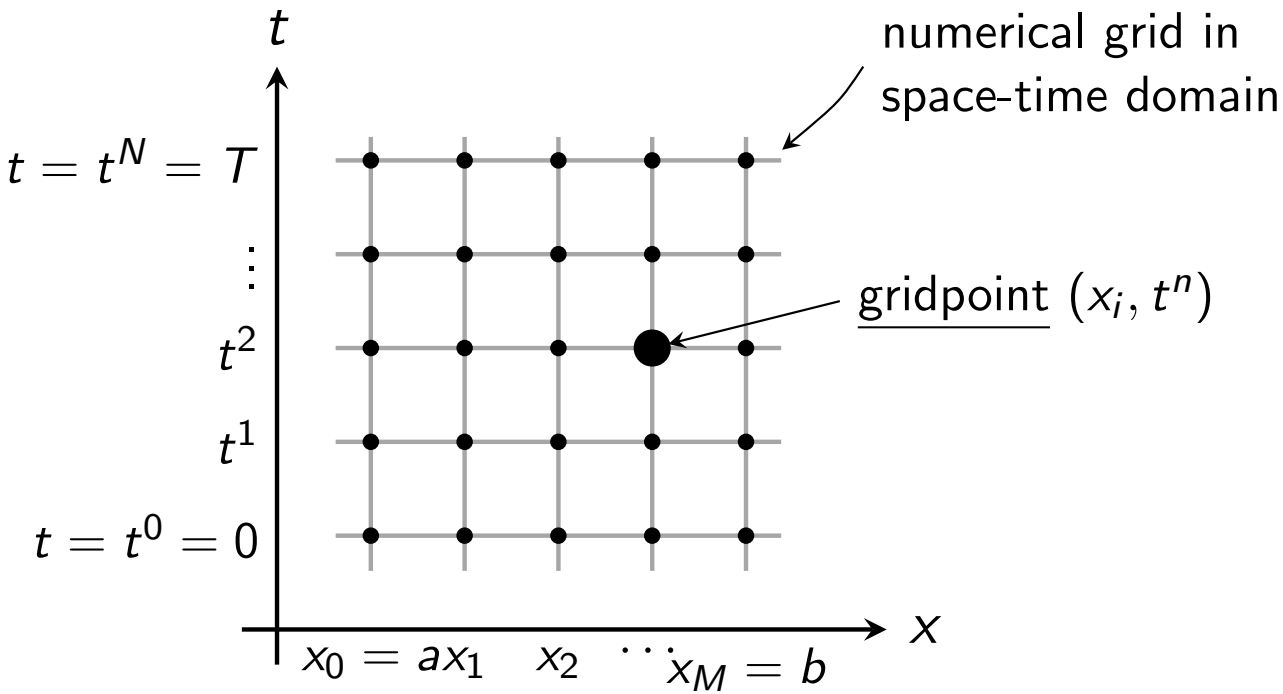
FTCS

↑ forward in time

↑ central in space

with $u_i^n \approx u(x_i, t^n)$

FTCS for heat equation [3]



FTCS for heat equation [4]

$$\text{FTCS}: \quad u_i^{n+1} = u_i^n + \frac{k\,\Delta t}{(\Delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

$$i = \ldots$$

$$n = \ldots$$

$$\text{IC}: \quad u_i^0 = \ldots \quad , \; i = \ldots$$

$$\text{BC's}: \quad \begin{cases} u_0^n = \ldots \quad , \; n = \ldots \\ u_M^n = \ldots \quad , \; n = \ldots \end{cases}$$

FTCS for heat equation [5]

Consider the general PDE: $\mathcal{L}u = 0$

and a FD-approximation: $L_\Delta u = 0$

Let $v(x,t)$ be any smooth function, then the local truncation error τ_i^n (LTE) is defined as:

$$\tau_i^n = \mathcal{L}\Big(v\big(i\,\underset{\uparrow}{\Delta x}_{x_i}\,,\; n\,\underset{\uparrow}{\Delta t}_{t^n}\big)\Big) - L_\Delta v(i\,\Delta x, n\,\Delta t)$$

FTCS for heat equation [6]

Example :

$$\begin{cases} \mathcal{L}v = v_t - k\, v_{xx} \quad \text{(heat equation)} \\ L_\Delta v = \dfrac{v_i^{n+1} - v_i^n}{\Delta t} - k \cdot \dfrac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{(\Delta x)^2} \quad \text{(FTCS)} \end{cases}$$

$$\implies \tau_i^n = \left[(v_t)_i^n - \frac{v_i^{n+1} - v_i^n}{\Delta t} \right] - k \cdot \left[(v_{xx})_i^n - \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{(\Delta x)^2} \right]$$

(using Taylor series)

$$= -\frac{\Delta t}{2} (v_{tt})_i^n + \frac{k\,(\Delta x)^2}{12} (v_{xxxx})_i^n + \text{H.O.T. in } \Delta t \text{ and } \Delta x$$

FTCS for heat equation [7]

Def. A FD-scheme $L_\Delta u_i^n = 0$ is consistent "with a PDE" $\mathcal{L}u = 0$ if $\tau_i^n \to 0$ as $\Delta x \to 0$ AND $\Delta t \to 0$

$\Longrightarrow$ **FTCS-scheme is consistent** (for the heat equation)

FTCS for heat equation [8]

FTCS for the heat equation can be re-written as:

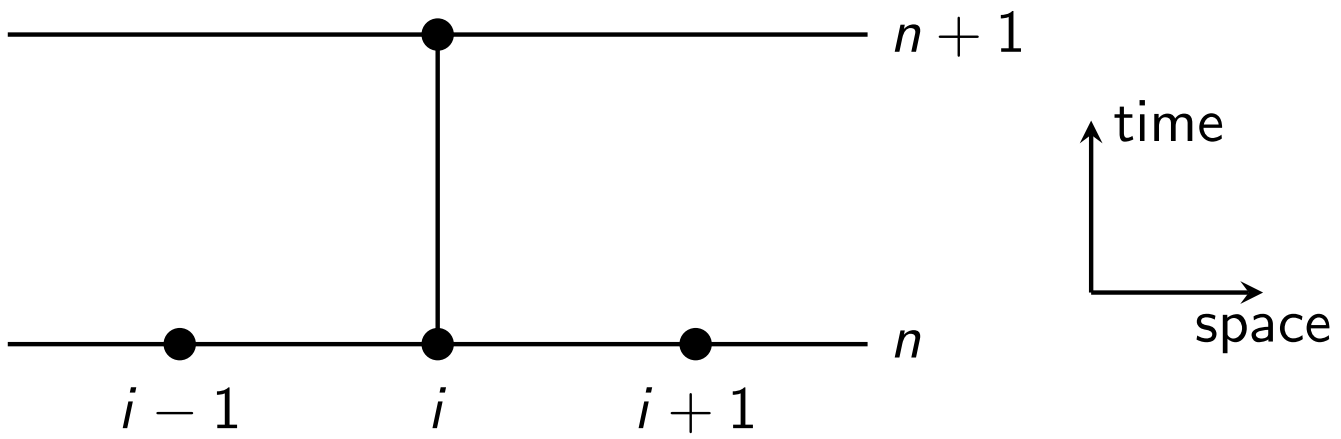
$$\boxed{\vec{u}^{n+1} = L_\Delta \vec{u}^n}$$

where $\vec{u}^n \stackrel{\text{def}}{=} \begin{pmatrix} \vdots \\ u_i^n \\ \vdots \end{pmatrix}$ and

$$L_\Delta = \begin{pmatrix} 1-2r & r & & 0 \\ r & 1-2r & r & \\ & \ddots & \ddots & \ddots \\ 0 & & r & 1-2r \end{pmatrix}, \qquad \boxed{r \stackrel{\text{def}}{=} k \cdot \frac{\Delta t}{(\Delta x)^2}}$$

FTCS for heat equation [9]

Computational stencil of FTCS (heat eq.) :



Note (using matrix norms): $\|L_\Delta\|_\infty = |1-2r| + 2r$

if $r \leq \frac{1}{2}$, then $1-2r \geq 0$ and $\boxed{\|L_\Delta\|_\infty = 1}$

FTCS for heat equation [10]

Def. : $\vec{u}^n = \begin{pmatrix} \vdots \\ u(x_i, t^n) \\ \vdots \end{pmatrix}$ "exact" values at grid points

---

Def. : A FD-approximation converges to the solution of a PDE on the interval $0 \leq t^n \leq T$, at $t = t^n$, if $\|\vec{u}^n - \vec{U}^n\| \to 0$

$$\text{for} \begin{cases} n \to \infty \\ \Delta x \downarrow 0 \\ \Delta t \downarrow 0 \\ t^n = n \cdot \Delta t \leq T \end{cases}$$

FTCS for heat equation [11]

Def. A FD-scheme is stable, if there exists a positive constant C such that: $\|\vec{U}^n\| \leq C \cdot \|\vec{U}^0\|$

$$\begin{array}{c} n\to\infty \\ \Delta x\to 0 \\ \Delta t\to 0 \\ \text{at } t^n = n\,\Delta t \leq T \end{array}$$

the constant C independent of the gridspacing and initial data

(a bound on the growth of the numerical solution)

$\sim$ well-posedness of the underlying PDE

FTCS for heat equation [12]

Thm: **FTCS for the heat equation converges in the maximum norm, if** $r \leq \frac{1}{2}$.

Proof: we have

$$U_i^{n+1} = r\,U_{i-1}^n + (1-2r)\,U_i^n + r\,U_{i+1}^n \quad \forall i \quad \text{(numerical solution)}$$

and

$$u_i^{n+1} = r\,u_{i-1}^n + (1-2r)\,u_i^n + r\,u_{i+1}^n + \Delta t \cdot \tau_i^n \quad \forall i \quad \text{(exact solution)}$$

let $e_i^n = u_i^n - U_i^n$ ("the error at (x_i, t^n)")

Because of the linearity of the problem $\Longrightarrow$

$$e_i^{n+1} = r\,e_{i-1}^n + (1-2r)\,e_i^n + r\,e_{i+1}^n + \Delta t \cdot \tau_i^n \quad \forall i$$

FTCS for heat equation [13]

Taking absolute values and using the triangle inequality
$(|a+b| \leq |a| + |b|)$

$$\begin{aligned} \implies |e_i^{n+1}| &\leq |r| \cdot |e_{i-1}^n| + |1-2r| \cdot |e_i^n| + |r| \cdot |e_{i+1}^n| + \Delta t \cdot |\tau_i^n| \\ &\leq (|r| + |1-2r| + |r|) \cdot \|\vec{e}^{\,n}\|_\infty + \Delta t \cdot \|\vec{\tau}^{\,n}\|_\infty \end{aligned}$$

we know that $r > 0$ and we assume $(*)$ $\underbrace{\overbrace{r \leq \frac{1}{2}} \implies |1-2r| \geq 0}_{\text{(stability condition)}}$

and the absolute value signs can be removed:
$|r| + |1-2r| + |r| = r + 1 - 2r + r = 1$

FTCS for heat equation [14]

$$\implies \quad |e_i^{n+1}| \leq \|\vec{e}^n\|_\infty + \Delta t \cdot \|\vec{\tau}^n\|_\infty \quad \forall i$$

and
$$\begin{aligned}
\|\vec{e}^{n+1}\|_\infty &\leq \|\vec{e}^n\|_\infty + \Delta t \cdot \|\vec{\tau}^n\|_\infty \\
&\leq \|\vec{e}^{n-1}\|_\infty + \Delta t \cdot \|\vec{\tau}^{n-1}\|_\infty + \Delta t \cdot \|\vec{\tau}^n\|_\infty \\
&\leq \quad \ldots \\
&\leq \|\vec{e}^0\|_\infty + n \cdot \Delta t \cdot \max_{\underbrace{0 \leq k \leq n}_{\stackrel{\text{def}}{=} \tau}} \|\vec{\tau}^k\|_\infty
\end{aligned}$$

FTCS for heat equation [15]

$$\left.\begin{array}{l}\text{Neglecting round-off errors at } t=0: \ \|\vec{e}^{\,0}\|_\infty = 0 \\ \text{note that } t^n = n\cdot\Delta t \le T \quad \forall \text{ combinations of } n \text{ and } \Delta t\end{array}\right\} \implies \|\vec{e}^{\,n}\|_\infty \le \underbrace{T}_{\text{constant}} \cdot \tau \quad (\circ\circ)$$

Remember: $\tau_i^n = -\frac{\Delta t}{2}(u_{tt})_i^n + \frac{k(\Delta x)^2}{12}(u_{xxxx})_i^n + \text{H.O.T.}$

$$\implies |\tau| \le \underbrace{\frac{\Delta t}{2}\max_{\substack{\forall t\\ \forall x}}|u_{tt}| + \frac{k(\Delta x)^2}{12}\max_{\substack{\forall t\\ \forall x}}|u_{xxxx}|}_{\to 0 \text{ for } \Delta t\to 0 \text{ and } \Delta x\to 0 \quad (\underline{\text{consistency}})} + \text{H.O.T.}\ \} \implies \tau \to 0$$

$(\circ\circ) \implies \|\vec{e}^{\,n}\|_\infty \to 0$ for Δx and $\Delta t \to 0$ ($\underline{\text{convergence}}$)

FTCS for heat equation [16]

Special case ("supra-convergence"):

FTCS heat eq. $\Big\}$ $\tau_i^n = \left(-\frac{1}{2}\,\Delta t\,k^2 + \frac{k}{12}(\Delta x)^2 \right) u_{xxxx} + \mathcal{O}\left((\Delta t)^2\right) + \mathcal{O}\left((\Delta x)^4\right)$

$u_t = k\,u_{xx}$
$u_{tt} = k\,(u_{xx})_t = k\,(u_t)_{xx} = k\,(u_{xx})_{xx} = k^2\,u_{xxxx}$

if we choose $\frac{\Delta t}{(\Delta x)^2} = \frac{1}{6k}$, then $\tau_i^n = \mathcal{O}\left((\Delta t)^2\right) + \mathcal{O}\left((\Delta x)^4\right)$

$\left(\Longrightarrow \frac{k\,\Delta t}{(\Delta x)^2} = \frac{1}{6} < \frac{1}{2} \right)$ $= \mathcal{O}\left((\Delta x)^4\right)$

fourth-order accurate !

Von Neumann stability [1]

Define :

A analytical solution of the PDE

D exact solution of the FD equation

N numerical solution using a computer with finite accuracy

$$\Longrightarrow \quad \text{discretization error} = A - D$$

$$\text{round-off error} \stackrel{\text{def}}{=} \varepsilon = N - D \quad \text{and} \quad N = D + \varepsilon$$

Von Neumann stability [2]

for FTCS applied to the heat equation with $k = 1$:

$$\frac{D_i^{n+1} + \varepsilon_i^{n+1} - D_i^n - \varepsilon_i^n}{\Delta t} = \frac{D_{i+1}^n + \varepsilon_{i+1}^n - 2D_i^n - 2\varepsilon_i^n + D_{i-1}^n + \varepsilon_{i-1}^n}{(\Delta x)^2}$$

(we have added a round-off error to each term in the exact FD equation)

D "must" solve the FTCS equation exactly

$$\Longrightarrow \quad \frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{(\Delta x)^2} \quad \boxed{\bullet\bullet}$$

so, ε satisfies the same FD equation !
(this is, because we consider a linear PDE)

Von Neumann stability [3]

Stability : the ε_i's do not grow from step n to step $n+1$ (in the time direction)

$$\Longleftrightarrow \quad |\varepsilon_i^{n+1}| \leq |\varepsilon_i^n| \quad \Longleftrightarrow \quad \left| \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} \right| \leq 1$$

Von Neumann stability [4]

Assume : the distribution of the errors "along the x-axis" can be given by a Fourier series in x, such that the time-wise variation is exponential in time, i.e. $\varepsilon(x,t) = e^{at} \sum_{m=1}^{\infty} e^{ik_m x}$

with $e^{at} \in \mathbb{C}$, k_m the wave number and $i = \sqrt{-1}$.

Von Neumann stability [5]

The FD equation is linear $\Longrightarrow$ each term in the series behaves in the same way as the series itself. Hence, let us deal with just one term of the series and write $\varepsilon_m(x,t) = e^{at}e^{ikmx}$ ◎

Substitute ◎ into ●● (previous page) :

$$t^n \leftrightarrow t\,,\; t^{n-1} \leftrightarrow t-\Delta t\,,\; t^{n+1} \leftrightarrow t+\Delta t$$
$$x_i \leftrightarrow x\,,\; x_{i-1} \leftrightarrow x-\Delta x\,,\; x_{i+1} \leftrightarrow x+\Delta x$$

$$\rightsquigarrow \quad \frac{e^{a(t+\Delta t)}\cdot e^{ikmx} - e^{at}\cdot e^{ikmx}}{\Delta t} = \frac{e^{at}e^{ikm(x+\Delta x)} - 2e^{at}e^{ikmx} + e^{at}e^{ikm(x-\Delta x)}}{(\Delta x)^2}$$

Von Neumann stability [6]

divide by $e^{at}e^{ik_m x}$ **:** $\quad \dfrac{e^{a\,\Delta t}-1}{\Delta t} = \dfrac{e^{ik_m\Delta x}-2+e^{-ik_m\Delta x}}{(\Delta x)^2}$

$$\text{or} \quad e^{a\,\Delta t} = 1 + \frac{\Delta t}{(\Delta x)^2}\left[e^{ik_m\Delta x}+e^{-ik_m\Delta x}-2\right]$$

remember
$$\cos(k_m\Delta x) = \frac{e^{ik_m\Delta x}+e^{-ik_m\Delta x}}{2} \quad \Longrightarrow \quad = 1 + \frac{2\,\Delta t}{(\Delta x)^2}\left[\cos(k_m\Delta x)-1\right]$$

remember
$$\sin^2\left(\frac{k_m\Delta x}{2}\right) = \frac{1-\cos(k_m\Delta x)}{2} \quad \Longrightarrow \quad = 1 - \frac{4\,\Delta t}{(\Delta x)^2}\sin^2\left(\frac{k_m\Delta x}{2}\right)$$

from ⊙ **on previous page ("dropping" the** m**) :**

$$\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = \frac{e^{a(t+\Delta t)}\cdot e^{ik_m x}}{e^{at}\cdot e^{ik_m x}} = e^{a\,\Delta t}$$

$$\Longrightarrow \quad \left|\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n}\right| = |e^{a\,\Delta t}| = \left|1 - \frac{4\,\Delta t}{(\Delta x)^2}\sin^2\left(\frac{k_m\Delta x}{2}\right)\right| \quad (a\in\mathbb{C})$$

Von Neumann stability [7]

$G = |e^{a\,\Delta t}|$ **is called the amplification factor**

stability for FTCS/heat if $G \leq 1$

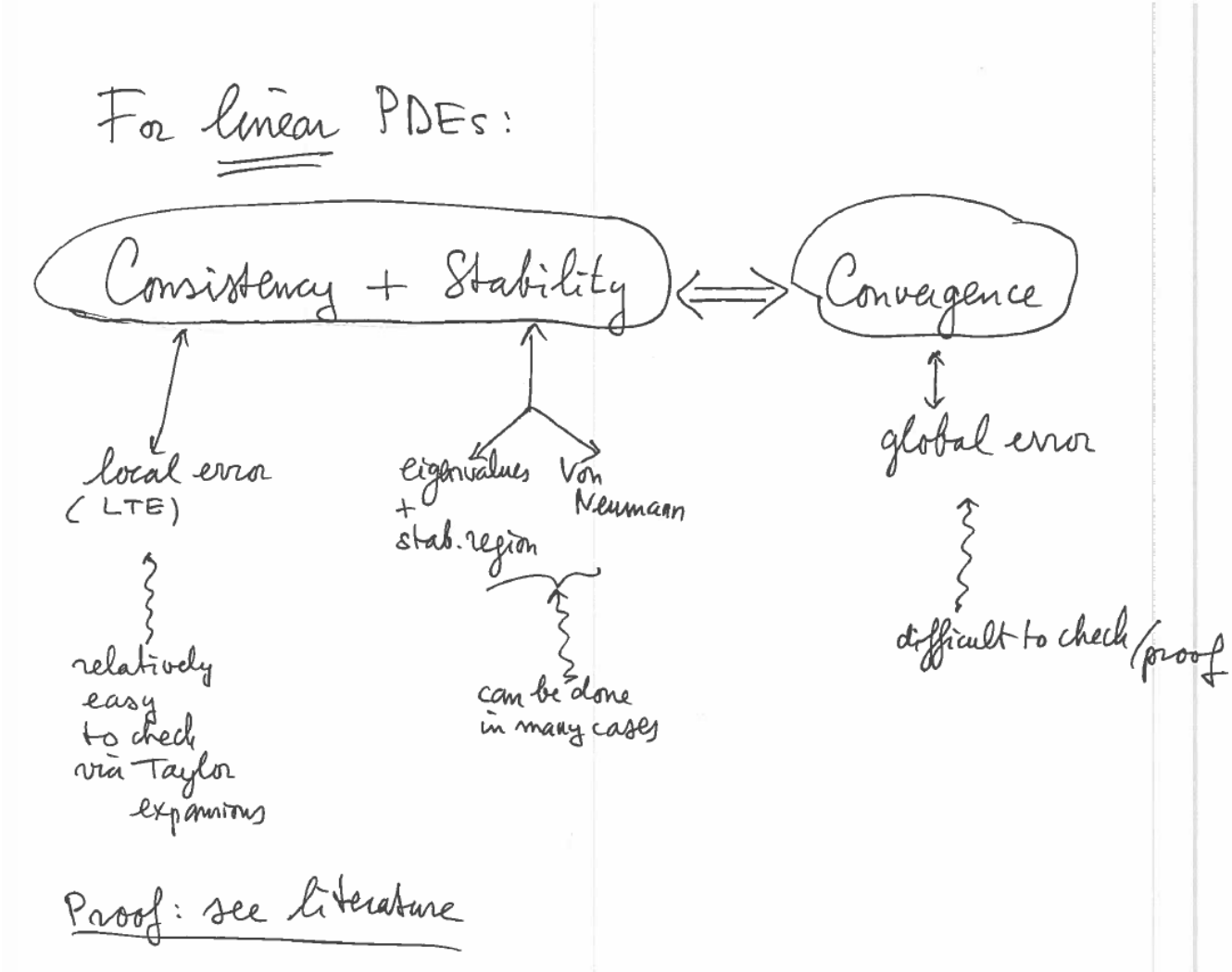
$$\iff \quad -1 \leq 1 - \frac{4\,\Delta t}{(\Delta x)^2} \sin^2\left(\frac{k_m \Delta x}{2}\right) \leq 1$$

1) $\dfrac{4\,\Delta t}{(\Delta x)^2} \sin^2\left(\dfrac{k_m \Delta x}{2}\right) \geq 0$ (always true) $\longleftrightarrow$ gives no extra information —

2) $\dfrac{\Delta t}{(\Delta x)^2} \underbrace{\sin^2\left(\dfrac{k_m \Delta x}{2}\right)}_{\in [0,1]} \leq \dfrac{1}{2}$

If $\underbrace{\dfrac{\Delta t}{(\Delta x)^2}}_{\substack{= r \\ (\text{with } k=1)}} \leq \dfrac{1}{2}$ **, then** $G \leq 1$

Lax equivalence theorem



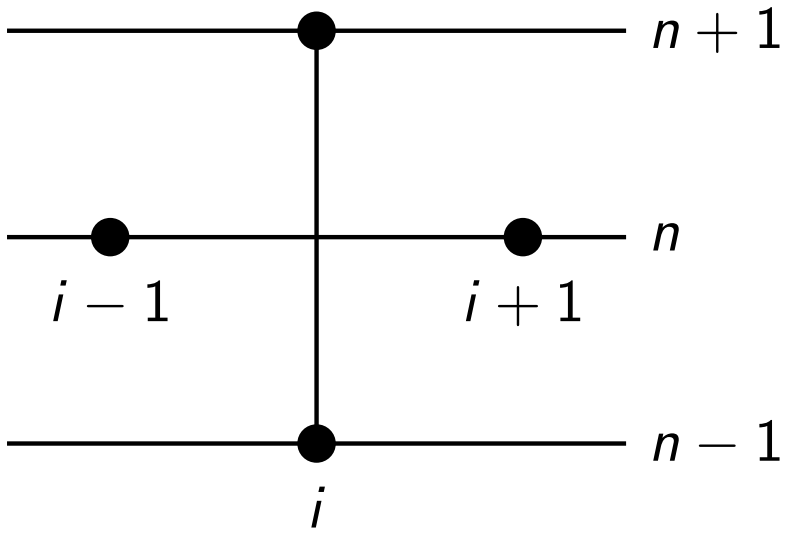
Conditional consistency [1]

Du Fort–Frankel scheme for the heat equation

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\,\Delta t} = k \cdot \frac{u_{i+1}^{n} - u_i^{n+1} - u_i^{n-1} + u_{i-1}^{n}}{(\Delta x)^2}$$

(not connected to the MoL approach!)

Computational stencil :



Conditional consistency [2]

LTE :

$$\tau_i^n = \frac{k}{12}(u_{xxxx})_i^n(\Delta x)^2 - (u_{tt})_i^n\left(\frac{\Delta t}{\Delta x}\right)^2 - \frac{1}{6}k^2(u_{ttt})_i^n(\Delta t)^2 + \ldots$$

IF we let Δt and Δx approach to zero at the same rate $\dfrac{\Delta t}{\Delta x} = \beta > 0$ (constant),
then the method is **inconsistent**.

In fact, it approximates the PDE: $u_t + \beta^2 u_{tt} = k \cdot u_{xx}$
("wave equation with damping")

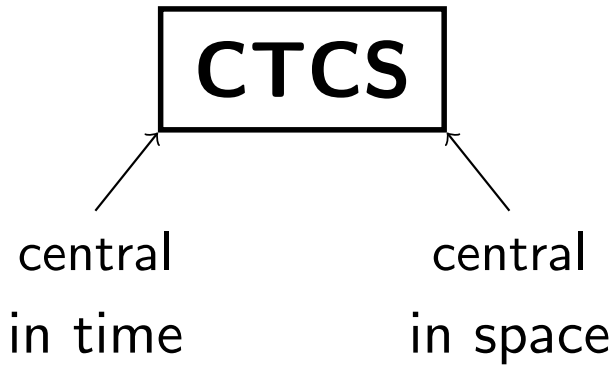
The scheme is called: **conditionally consistent**

Unconditional instability [1]

Leapfrog scheme for the heat equation ("haasje-over")

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\,\Delta t} = \frac{k}{(\Delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

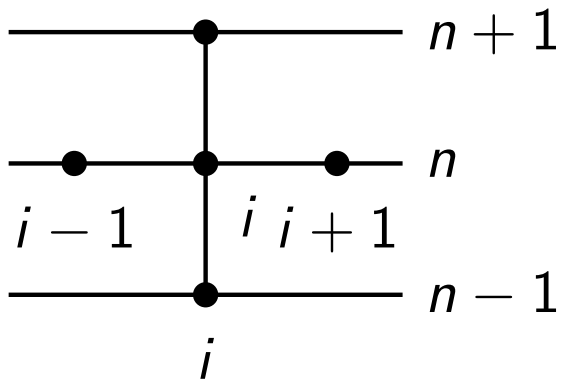
(in the MoL sense / it is explicit midpoint / for the central in space / discretization)



- A three-level method
  $\rightarrow n-1, n, n+1 \implies$ requires extra starting value
- Local Truncation Error (LTE):
  $\tau_i^n = \mathcal{O}\left((\Delta t)^2\right) + \mathcal{O}\left((\Delta x)^2\right)$
  $\implies$ **Consistent**

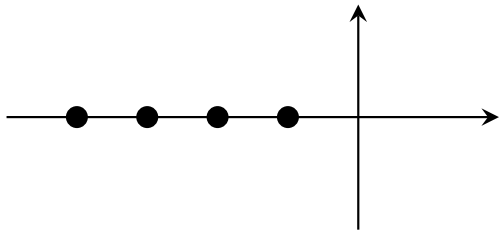
Unconditional instability [2]

Computational stencil:

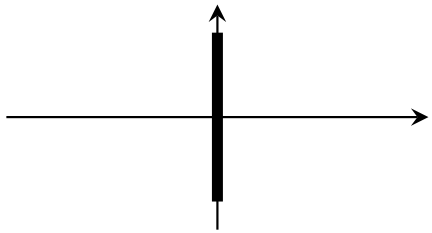


Note that $\lambda(D_{2c}) \in \mathbb{R}^-$ **(see before) and** $S \in i\mathbb{R}$ **(on the imaginary axis in the complex plane)**

explicit midpoint



and



GTCS for ... conditionally ... variable

Unconditional instability [3]

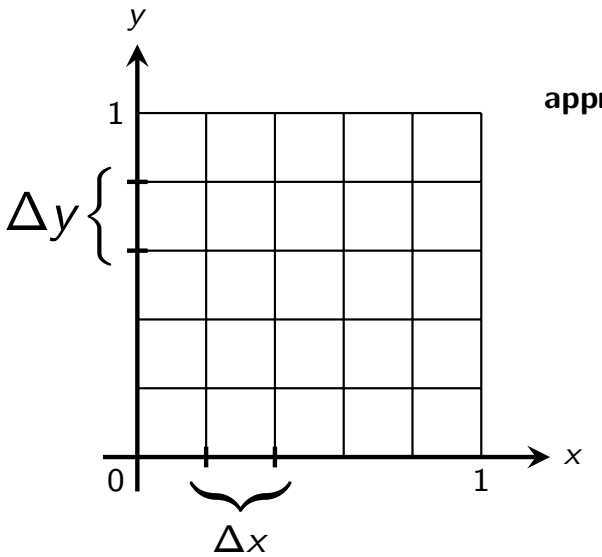
$\Longrightarrow$ **CTCS for heat equation unconditionally unstable!!!**

(for all $\Delta t, \Delta x$ choices)

The heat equation in 2d [1]

$$u_t = k \cdot (u_{xx} + u_{yy}) = k \cdot \Delta u \ , \quad \Omega = [0,1] \times [0,1]$$

$$u(x, y, 0) = u_0(x, y) + \text{BC's on } \partial\Omega$$



approximation of Δu **in** (x_i, y_j) **:**

$$u_{xx}|_{(x_i,y_j)} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} + \ldots$$

$$u_{yy}|_{(x_i,y_j)} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2} + \ldots$$

Take, for example, $\Delta x = \Delta y = h$ **:**

$$\Delta u|_{(x_i,y_j)} = \frac{1}{h^2}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}\right) - \frac{h^2}{12}\left[u_{xxxx}|_{i,j} + u_{yyyy}|_{i,j}\right] + \ldots$$

The heat equation in 2d [2]

Time–integration :

Choice 1 **EF**

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{k\,\Delta t}{h^2}\left[u_{i-1,j}^{n} + u_{i+1,j}^{n} + u_{i,j-1}^{n} + u_{i,j+1}^{n} - 4u_{i,j}^{n}\right]$$

$$\text{LTE}: \ \mathcal{O}(h^2) + \mathcal{O}(\Delta t)$$

Stability: substitute (Von Neumann stability) $u_{i,j}^{n} = e^{at}e^{i\xi x}e^{i\eta y}$

$\Longrightarrow$ amplification factor:

$$G = \left|1 - 4\,\frac{k\,\Delta t}{h^2}\sin^2\left(\frac{\xi h}{2}\right) - 4\,\frac{k\,\Delta t}{h^2}\sin^2\left(\frac{\eta h}{2}\right)\right|$$

it follows that $G \leq 1$, if $0 \leq \dfrac{k\,\Delta t}{h^2} \leq \dfrac{1}{4}$

(factor two smaller than in 1D)

$\Longrightarrow$ **conditionally stable** & 1^{st} order accuracy in time direction

The heat equation in 2d [3]

Choice 2 **Trapezoidal method in time**

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = k \cdot \frac{\Delta_x^2 u_{i,j}^{n} + \Delta_x^2 u_{i,j}^{n+1}}{2(\Delta x)^2} + k \cdot \frac{\Delta_y^2 u_{i,j}^{n} + \Delta_y^2 u_{i,j}^{n+1}}{2(\Delta y)^2}$$

short notation for FD approx.
(see also lecture 2)

$$\Delta_x^2 u_i = u_{i+1} - 2u_i + u_{i-1}$$
$$\Delta_y^2 u_j = u_{j+1} - 2u_j + u_{j-1}$$

The heat equation in 2d [4]

$\Longrightarrow$ **Crank-Nicolson method** **for the 2D heat equation**

LTE : $\mathcal{O}\left((\Delta x)^2\right) + \mathcal{O}\left((\Delta y)^2\right) + \mathcal{O}\left((\Delta t)^2\right)$

stability : $G \leq 1 \quad \forall\, \xi, \eta \quad$ (unconditionally stable)

However, re-writing the FD equations, gives in each timestep :

sparse matrix $\longrightarrow$ $A\,\vec{u}^{n+1} = \vec{u}^{n}$, but no longer, as in 1D,

(with a simple tridiagonal structure)

$\Longrightarrow$ **stable, but "expensive"**

The heat equation in 2d [5]

Choice 3 Alternating-Direction scheme (ADI)

"in one direction (x) implicit, in the other direction (y) explicit" and also the other way round" and "combine both directions"

This is a splitting method.
(See also later on in the course)

The heat equation in 2d [6]

i) $$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\frac{1}{2}\Delta t} = \frac{k}{(\Delta x)^2}\Delta_x^2 u_{i,j}^{n+\frac{1}{2}} + \frac{k}{(\Delta y)^2}\Delta_y^2 u_{i,j}^{n}$$

half time step

solve tridiagonal system (conditionally stable!? $k \cdot \frac{\Delta t/2}{(\Delta y)^2} \leq \frac{1}{2}$)

ii) **take another half time step :**

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} = \frac{k}{(\Delta x)^2}\Delta_x^2 u_{i,j}^{n+\frac{1}{2}} + \frac{k}{(\Delta y)^2}\Delta_y^2 u_{i,j}^{n+1}$$

solve tridiagonal system (stability restriction: $k \cdot \frac{\Delta t/2}{(\Delta x)^2} \leq \frac{1}{2}$)

However, the combined method of i) and ii) (full time step):

$$\frac{1}{2}\left(\Delta_x^2 + \Delta_y^2\right)\left(u_{i,j}^{n+1} + u_{i,j}^{n}\right) = \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + \frac{\Delta t}{4}\Delta_x^2\Delta_y^2\left(u_{i,j}^{n+1} - u_{i,j}^{n}\right)$$

is UNconditionally stable ! (see literature)

error: $\mathcal{O}\left((\Delta t)^2\right) + \mathcal{O}\left((\Delta x)^2\right) + \mathcal{O}\left((\Delta y)^2\right)$

efficiency : two tridiagonal systems to be solved

Outlook to Lecture 5

- prepare exercises of Lecture 4 (see webpage!)
- the advection equation
- FTCS
- upwind, downwind
- Lax-Friedrichs
- Lax-Wendroff