Lecture 5

Paul Andries Zegeling

Department of Mathematics, Utrecht University

Numerical Methods for PDEs

Outline of Lecture 5

- exercises of Lecture 4
- the advection equation: travelling wave solutions
- FTCS
- Lax-Friedrichs & Lax-Wendroff
- upwind/downwind & Beam-Warming
- modified PDE/equation & CFL-condition
- outlook to Lecture 6

The advection equation [1]

$$\boxed{\begin{aligned} u_t + cu_x &= 0 \\ u(x,0) &= u_0(x) \end{aligned}} \quad \Rightarrow \quad u(x,t) = u_0(x - ct)$$

1) Method of Characteristics:

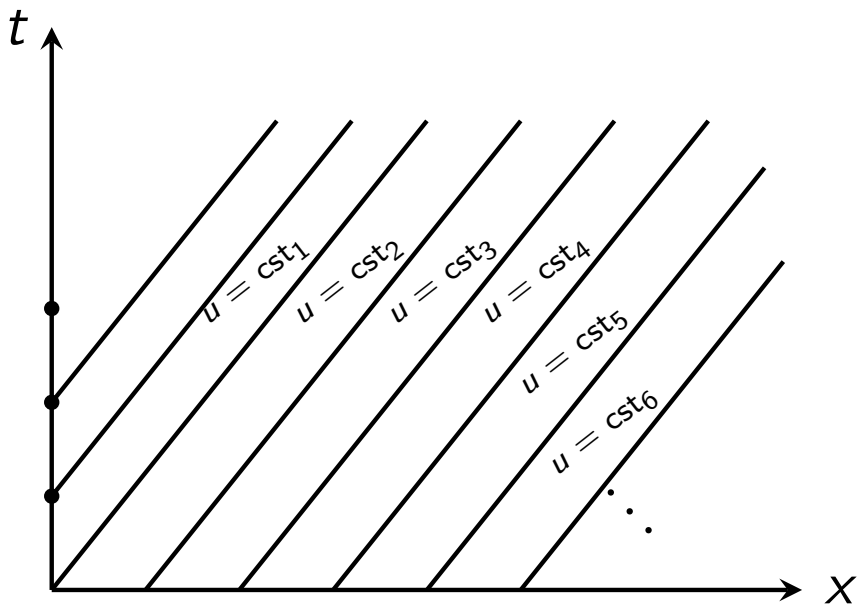
$$\underbrace{\frac{dt}{ds} = 1 \quad , \quad \frac{dx}{ds} = c}_{\text{characteristics}} \quad , \quad \frac{du}{ds} = 0$$

$$\Rightarrow \quad \underbrace{\frac{du}{dt}}_{\text{total derivative}} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \underbrace{\frac{dx}{dt}}_{= \frac{dx}{ds} \Big/ \frac{dt}{ds} = c} = 0 \quad \Rightarrow$$

u = constant on each characteristic curve:

$x - ct = C$

The advection equation [2]



The advection equation [3]

2) "energy" : define $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}(u(x,t))^2\,dx$

advection PDE $*u$:

$$\begin{aligned} uu_t + cuu_x &= 0 \\ \Leftrightarrow \quad uu_t &= -cuu_x \\ \Leftrightarrow \quad \frac{1}{2}(u^2)_t &= -\frac{c}{2}(u^2)_x \\ \Leftrightarrow \quad \frac{1}{2}\int_{-\infty}^{\infty}(u^2)_t\,dx &= -\frac{c}{2}\int_{-\infty}^{\infty}(u^2)_x\,dx \end{aligned}$$

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\left[\frac{1}{2}\int_{-\infty}^{\infty}(u(x,t))^2\,dx\right] \\ &= \frac{1}{2}\int_{-\infty}^{\infty}(u^2)_t\,dx \\ &= -\frac{c}{2}\cdot\lim_{L\to\infty}\int_{-L}^{L}(u^2)_x\,dx \\ &= -\frac{c}{2}\cdot\lim_{L\to\infty}\left[u^2(L,t)-u^2(-L,t)\right] \\ &= 0 \quad (\text{assume } \lim_{L\to\infty}u(\pm L,t)=0) \end{aligned}$$

The advection equation [4]

$$\Rightarrow \frac{dE}{dt} = 0 \quad \text{and} \quad E(t) = E(0) = \frac{1}{2}\int_{-\infty}^{\infty} (u(x,0))^2\, dx = \frac{1}{2}\int_{-\infty}^{\infty} (u_0(x))^2\, dx \quad \forall t \geq 0$$

$\Rightarrow$ **“Energy” is conserved**

The advection equation [5]

3) $\underbrace{\text{Fourier-transform method}}_{\text{"in the } x\text{-direction"}}$ (see Lecture 1) $\quad\Rightarrow\quad u(x,t) = u_0(x-ct)$

4) $\underbrace{\text{Laplace-transform method}}_{\text{"in the } t\text{-direction"}}$

take $c = 1$ for example $\quad\underbrace{\Rightarrow}_{\text{apply } \mathcal{L}\text{-transform}}$

$$s\,U(x,s) - u_0(x) + U_x(x,s) = 0 \quad \text{with } U(x,s) = \mathcal{L}(u(x,t))$$

solve ODE $\Rightarrow$

$$\begin{aligned} U(x,s) &= e^{-sx}\int_0^x e^{sz}\,u_0(z)\,dz \\ &= \int_0^x u_0(z)\,e^{-(x-z)s}\,dz \end{aligned}$$

apply $\mathcal{L}^{-1}$ $\Rightarrow$

$$u(x,t) = \int_0^x u_0(z)\left[\mathcal{L}^{-1}\left(e^{-(x-z)s}\right)_{s\to t}\right]dz$$

$$\underbrace{=}_{\substack{\text{property}\\ \text{of } \mathcal{L}\text{-trafo}}} \int_0^x u_0(z)\cdot\delta(\underbrace{t-(x-z)}_{=z-(x-t)})\,dz = u_0(x-t) \quad (\text{"shifting property"})$$

The advection equation [6]

5) travelling wave assumption ("TW Ansatz")

Find $u(x,t)$ of the form $\varphi(\underbrace{x-vt}_{=\xi \text{ (TW-coordinate)}})$ with constant speed v

$$u_t = -v\varphi' \quad \& \quad u_x = 1 \cdot \varphi'$$

where $'$ denotes differentiation with respect to ξ.

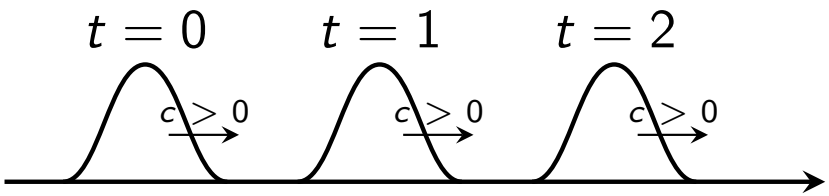
$$\Rightarrow -v\varphi' + c\varphi' = 0 \quad \overset{\varphi' \not\equiv 0}{\Rightarrow} \quad v = c \quad \text{and} \quad u(x,t) = \varphi(x-ct)$$

with $u(x,0) = u_0(x) \Rightarrow u(x,t) = u_0(x-ct)$

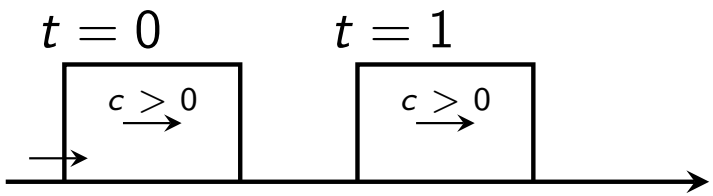
This is a general concept, also applicable to nonlinear PDEs (here, it seems rather trivial).

The advection equation [7]

smooth solutions



discontinuous solutions



FTCS [1]

FTCS applied to heat equation $\rightsquigarrow$ stability constraint $\dfrac{k \cdot \Delta t}{(\Delta x)^2} \leq \dfrac{1}{2}$

FTCS applied to advection equation $\rightsquigarrow$ " $\dfrac{\Delta t}{\Delta x} \leq C$ " **?**

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0$$

$$\Rightarrow \quad u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)$$

FTCS [2]

Taylor expansions:

$$(u_t)_i^n+(cu_x)_i^n+\frac{\Delta t}{2}(u_{tt})_i^n+c\cdot\frac{(\Delta x)^2}{6}(u_{xxx})_i^n+\mathcal{O}((\Delta t)^2)+\mathcal{O}((\Delta x)^4)=0$$

LTE =

$$\tau_i^n=\underbrace{\frac{\Delta t}{2}(u_{tt})_i^n}_{\text{first order in time}}+\underbrace{c\frac{(\Delta x)^2}{6}(u_{xxx})_i^n}_{\text{second order in space}}+\mathcal{O}((\Delta t)^2)+\mathcal{O}((\Delta x)^4)$$

as $\Delta t, \Delta x \to 0 \quad \Rightarrow \quad \tau_i^n \to 0 \quad \Rightarrow$

FTCS for advection equation is consistent

FTCS [3]

Von Neumann stability analysis: insert $u_i^n = e^{at} \cdot e^{i\xi_m x} \quad (i = \sqrt{-1})$

$$\Rightarrow \frac{e^{a(t+\Delta t)} e^{i\xi_m x} - e^{at} e^{i\xi_m x}}{\Delta t} + \frac{c}{2\Delta x} \left[e^{at} e^{i\xi_m (x+\Delta x)} - e^{at} e^{i\xi_m (x-\Delta x)} \right] = 0$$

divide by $e^{at} e^{i\xi_m x}$ $\Rightarrow$ $\frac{e^{a\Delta t} - 1}{\Delta t} + \frac{c}{2\Delta x} \left[e^{i\xi_m \Delta x} - e^{-i\xi_m \Delta x} \right] = 0$

$$\Rightarrow \text{amplification factor: } G = e^{a\Delta t} = 1 - i \frac{c\Delta t}{\Delta x} \sin(\xi_m \Delta x)$$

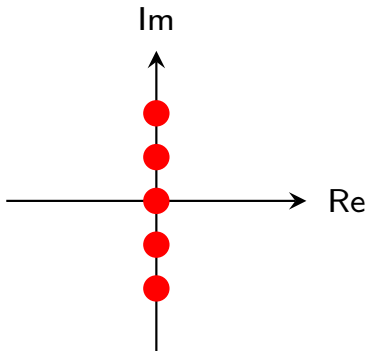
$$\text{and } |G| = \sqrt{1 + \left(\frac{c\Delta t}{\Delta x} \right)^2 \sin^2(\xi_m \Delta x)} > 1 \quad \Rightarrow \quad \underline{\textbf{unconditionally unstable}}$$

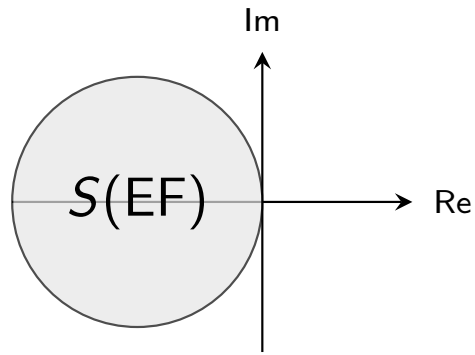
(for any Δt, Δx combination)

FTCS [4]

Two other ways of interpreting this instability

1) the spectrum $\sigma(D_{1c}) \in i\mathbb{R}$ and $\underbrace{S(\text{EF})}_{\text{stability region of EF}}$





$$\Rightarrow \quad \sigma(D_{1c}) \cap S(\text{EF}) = \emptyset$$

FTCS [5]

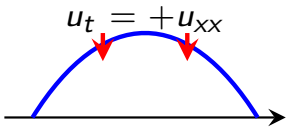
2) re-write the advection equation and its LTE:

$$u_t + cu_x = 0 \quad \Rightarrow \quad (u_t)_t + (cu_x)_t = u_{tt} + cu_{xt} = u_{tt} + c\left(-cu_x\right)_x$$
$$= u_{tt} - c^2 u_{xx} = 0$$
$$\Rightarrow \quad u_{tt} = c^2 u_{xx}$$

LTE:

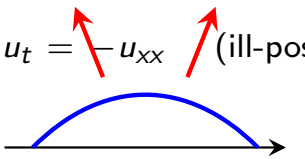
$$(u_t)_i^n + (cu_x)_i^n = -\frac{\Delta t}{2}(u_{tt})_i^n + \ldots$$
$$= \underbrace{-\frac{\Delta t}{2}c^2(u_{xx})_i^n + \ldots}_{\text{“negative diffusion” (unstable)}}$$

remember



(damping)

&



Lax-Friedrichs [1]

A minor modification to FTCS:

replace u_i^n by an averaged value $\frac{1}{2}(u_{i-1}^n + u_{i+1}^n)$

$$\Rightarrow \quad u_i^{n+1} = \frac{1}{2}(u_{i-1}^n + u_{i+1}^n) - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

is called Lax-Friedrichs method ("LF")

Note: $\frac{1}{2}(u_{i-1}^n + u_{i+1}^n) = u_i^n + \frac{1}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$

$$\overset{\text{LF}}{\Rightarrow} \quad u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{1}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

Lax-Friedrichs [2]

re-arrange terms:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \frac{(\Delta x)^2}{2\Delta t}\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

Calculate LTE via Taylor expansions $\Rightarrow$ a consistent approximation of $u_t + cu_x = 0$ for $\Delta t, \Delta x \to 0$

Lax-Friedrichs [3]

However, it looks more as a discretization of the advection-diffusion PDE:

$$u_t + cu_x = \underbrace{\varepsilon_{\text{LF}}}_{\substack{\text{“numerical} \\ \text{diffusion”} \text{ term}}} u_{xx} \quad \text{with } \varepsilon_{\text{LF}} = \frac{(\Delta x)^2}{2\Delta t}$$

(‘numerical damping” of the solution)

Stable?

Lax-Friedrichs [4]

Note that

$$\text{Lax-Friedrichs method} = \underbrace{\text{FTCS}}_{\text{unstable}} + \underbrace{\text{“numerical diffusion”}}_{\text{stabilizing? effect}}$$

} in the Method of Lines sense

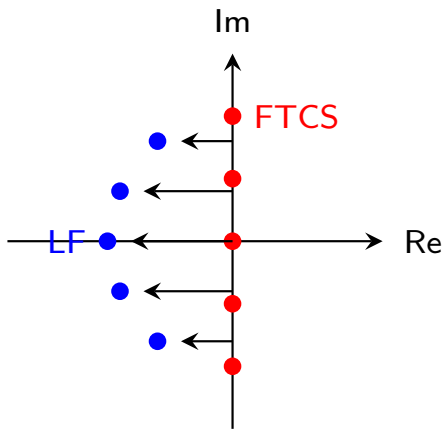
or LF = EF applied to the ODE system $\dot{\vec{u}} = B_{\text{LF}}\vec{u}$

$$\text{with } B_{\text{LF}} = \underbrace{-\frac{c}{2\Delta x}\begin{pmatrix} \ddots & \ddots & \theta \\ -1 & 0 & 1 \\ \theta & \ddots & \ddots \end{pmatrix}}_{\substack{\text{skew-symmetric} \\ \text{with } \lambda\text{'s} \in i\mathbb{R}}} + \underbrace{\frac{\varepsilon_{\text{LF}}}{(\Delta x)^2}\begin{pmatrix} \ddots & \ddots & \theta \\ 1 & -2 & 1 \\ \theta & \ddots & \ddots \end{pmatrix}}_{\substack{\text{symmetric} \\ \text{with } \lambda\text{'s} \in \mathbb{R}^-}}$$

Lax-Friedrichs [5]

The eigenvectors of both matrices are the same $\Rightarrow$

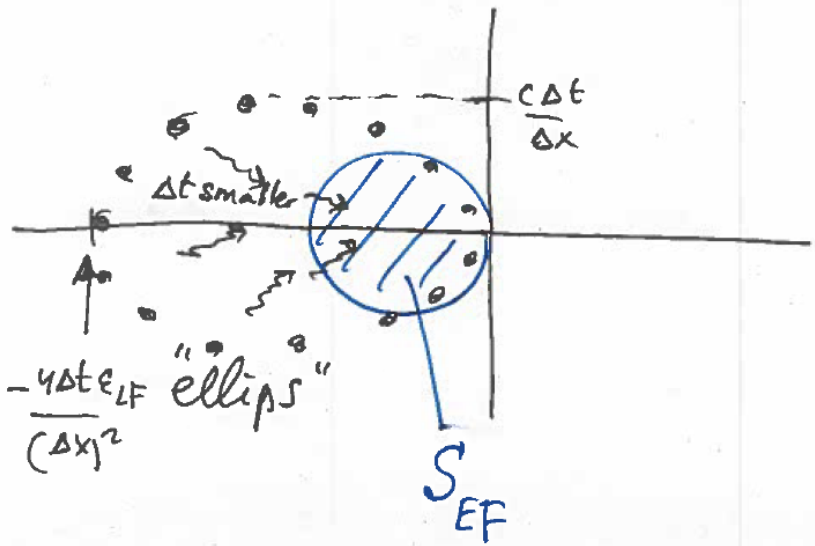
$$\lambda_p(B_{\text{LF}}) = -\frac{ic}{\Delta x}\sin(2\pi p\Delta x) - \frac{2\varepsilon_{\text{LF}}}{(\Delta x)^2}\big(1-\cos(2\pi p\Delta x)\big)$$



"The eigenvalues of the D_{1c}-matrix for u_x are shifted off the imaginary axis to the left half plane $\mathbb{C}^-$"

Lax-Friedrichs [6]

Stability is now possible (under certain restrictions)



it can be shown that

LF is stable, if $\left|\left|\frac{c\,\Delta t}{\Delta x}\right|\right| \leq 1$

$\Leftrightarrow$

$\Delta t \cdot \sigma(B_{LF}) \in S_{EF}$

Lax-Wendroff [1]

A higher order approximation $\mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta x)^2)$ instead of $\mathcal{O}(\Delta t) + \mathcal{O}((\Delta x)^2)$

Taylor: $u(x, t + \Delta t) = u(x, t) + \Delta t\, u_t(x, t) + \frac{(\Delta t)^2}{2} u_{tt}(x, t) + \ldots$
(in t-direction)

replace u_t by $-cu_x$ and u_{tt} by $+c^2 u_{xx}$

$$\Rightarrow \quad u(x, t+\Delta t) = u(x, t) - c\Delta t\, u_x(x, t) + \frac{1}{2} c^2 (\Delta t)^2 u_{xx}(x, t) + \ldots$$

Lax-Wendroff [2]

use $u_x(x_i,t) \approx \underbrace{\frac{u_{i+1}-u_{i-1}}{2\Delta x}}_{\sim D_{1c}}$ and $u_{xx}(x_i,t) \approx \underbrace{\frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta x)^2}}_{\sim D_{2c}(=D_{1+}D_{1-})}$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{“central approximations”}}$

and EF in the time-direction

$$\Rightarrow \quad u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{(\Delta t)^2}{2}c^2\frac{(u_{i+1}^n - 2u_i^n + u_{i-1}^n)}{(\Delta x)^2}$$

(still a three-point stencil!) $\Rightarrow$ Lax-Wendroff method (LW)

Lax-Wendroff [3]

Write this as an ODE-system (in the MoL-sense):

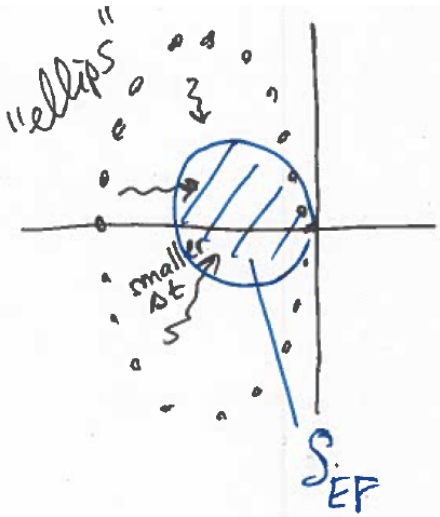
$$\dot{\vec{u}} = B_{\text{LW}}\vec{u} \quad (+ \text{ EF})$$

$$\text{with } B_{\text{LW}} = -\frac{c}{2\Delta x}\begin{pmatrix} \ddots & \ddots & \theta \\ -1 & 0 & 1 \\ \theta & \ddots & \ddots \end{pmatrix} + \frac{\varepsilon_{\text{LW}}}{(\Delta x)^2}\begin{pmatrix} \ddots & \ddots & \theta \\ 1 & -2 & 1 \\ \theta & \ddots & \ddots \end{pmatrix} \quad \text{and } \varepsilon_{\text{LW}} = \frac{c^2\Delta t}{2}$$

Note that

$$\Delta t \cdot \lambda_p(B_{\text{LW}}) = -i\frac{c\Delta t}{\Delta x}\sin(2\pi p\Delta x) - \left(\frac{c\Delta t}{\Delta x}\right)^2\left(1-\cos(2\pi p\Delta x)\right)$$

Lax-Wendroff [4]



ellipses with centre at $-\left(\frac{c\Delta t}{\Delta x}\right)^2$ and semi-axis $\left(\frac{c\Delta t}{\Delta x}\right)^2$ and $\frac{c\Delta t}{\Delta x}$

Stability, if $\left|\frac{c\Delta t}{\Delta x}\right| \leq 1$ (then $\Delta t \lambda_p(B_{LW}) \in S_{EF}$)

Accuracy: 2nd order in space and time

Upwind-downwind [1]

One-sided (non-symmetric) approximations:

$$u_x(x_i,t) \approx \frac{u_i - u_{i-1}}{\Delta x} \quad \text{and} \quad u_x(x_i,t) \approx \frac{u_{i+1} - u_i}{\Delta x}$$

coupled with EF for u_t:

$$\text{FTBS} \quad u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$$

$\uparrow$ backward

$$\text{FTFS} \quad u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}(u_{i+1}^n - u_i^n)$$

$\uparrow$ forward

Both methods are $\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)$ "first order"

Upwind-downwind [2]

Note that the exact solution reads: $u(x,t) = u_0(x - ct)$

at the grid point (x_i, t^{n+1}):
$u(x_i, t + \Delta t) = u_0((x_i - c\Delta t) - ct) = u(x_i - c\Delta t, t)$

$\Rightarrow$ the solution at x_i on the next time level is given by data to the left of x_i if $c > 0$ and by data to the right of x_i if $c < 0$

this suggests that FTBS might be a better choice for $c > 0$ and FTFS for $c < 0$.

Upwind-downwind [3]

Von Neumann stability analysis shows:
that FTBS is stable

$$\text{only if } 0 < \frac{c\Delta t}{\Delta x} < 1$$

Since $\Delta t, \Delta x > 0$, FTBS can only be used for $c > 0$

and FTFS is stable only if $-1 < \frac{c\Delta t}{\Delta x} < 0$

$$\Rightarrow \text{FTFS for } c < 0$$

Upwind-downwind [4]

FTBS can be written as (compare Lax-Friedrichs)

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{c\Delta t\Delta x}{2}\frac{(u_{i+1}^n - 2u_i^n + u_{i-1}^n)}{(\Delta x)^2}$$

$$\mathcal{E}_{\text{upw}} \downarrow = \frac{c\Delta x}{2} \qquad \downarrow$$

$$\approx u_{xx}(x_i, t^n)$$

Note that, for $c < 0$: $\mathcal{E}_{\text{upw}} < 0$.
Negative diffusion added to the FD-scheme:

$$\Rightarrow \begin{array}{l} \text{unstable for } c < 0 \\ \text{and stable (conditionally) for } c > 0 \end{array}$$

Upwind-downwind [5]

FTFS can be re-written similarly but now with $\mathcal{E}_{\text{upw}} = -\frac{c\Delta x}{2}$

$$\Rightarrow \text{FTFS is unstable for } c > 0$$

and conditionally stable for $c < 0$

The numerical diffusion constants read:

$$\mathcal{E}_{\text{LF}} = \frac{\Delta x}{2\Delta t}$$

$$\mathcal{E}_{\text{LW}} = \frac{c^2 \Delta t}{2}$$

$$\mathcal{E}_{\text{upw}} = \frac{\text{sign}(c)\,\Delta x}{2}$$

Upwind-downwind [6]

A 2nd order upwind method (one-sided approximation):

follow the derivation of Lax-Wendroff and use one-sided FD's for the spatial derivatives: (for $c > 0$)

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}\left(3u_i^n - 4u_{i-1}^n + u_{i-2}^n\right) + \frac{1}{2}c^2\left(\frac{\Delta t}{\Delta x}\right)^2\left(u_i^n - 2u_{i-1}^n + u_{i-2}^n\right)$$

"Beam-Warming" method (BW)

Upwind-downwind [7]

BW is stable (via von Neumann) for $0 < \frac{c\Delta t}{\Delta x} < 2 \quad (c > 0)$

Theorem 1 (Lax-Richtmyer equivalence theorem)

← see also previous lecture

Given a (well-posed) linear hyperbolic PDE and its FD approximation that satisfies the consistency condition. Then stability is the necessary and sufficient condition for convergence of the method.

Theorem 2 (see literature for a proof)

There exists NO explicit unconditionally stable FD scheme for solving hyperbolic PDEs.

Theorem 3 Satisfying the so-called "CFL-condition" ⟶ see later

is necessary for convergence (but not sufficient!)

Modified equation [1]

We may ask ourselves now the following question:

Is there a PDE $v_t = \ldots$ such that our current FD approximation for $u_t + cu_x = 0$ at (x_i, t^n), u_i^n, is actually the "exact" solution for this new PDE, i.e., $u_i^n = v(x_i, t^n)$?

OR, a bit less ambitiously, can we, at least, find a PDE $v_t = \ldots$, let's call it new PDE, that is "better" satisfied by u_i^n than it does for the original PDE?

(Then, studying the behavior of $\widetilde{\text{PDE}}$ tells us much about the numerical approximation of our original PDE)

Example: upwind method for $c > 0$

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) \qquad \text{(FTBS)}$$

1) insert $v(x, t)$ into the FD equation

Modified equation [2]

2) suppose $v(x,t)$ agrees exactly with u_i^n at the grid points

$\Rightarrow$ $v(x,t)$ satisfies FTBS line-equation exactly.

(unlike $u(x,t)$!!!)

$$\underbrace{v(x,t+\Delta t)}_{u_i^{n+1}} = \underbrace{v(x,t)}_{u_i^n} - \frac{c\Delta t}{\Delta x}\left(\underbrace{v(x,t)}_{u_i^n} - \underbrace{v(x-\Delta x,t)}_{u_{i-1}^n}\right)$$

3) expand these terms in Taylor series about (x,t):

$$\begin{cases} v(x,t+\Delta t) = v(x,t) + \Delta t\, v_t + \frac{1}{2}(\Delta t)^2 v_{tt} + \dots \\ \\ v(x-\Delta x,t) = v(x,t) - \Delta x\, v_x + \frac{1}{2}(\Delta x)^2 v_{xx} + \dots \end{cases}$$

$$\begin{aligned} \Rightarrow \quad 0 &= v(x,t+\Delta t) - v(x,t) + \frac{c\Delta t}{\Delta x}\left(v(x,t) - v(x-\Delta x,t)\right) \\ &= \Delta t\, v_t + \frac{1}{2}(\Delta t)^2 v_{tt} + \dots + \frac{c\Delta t}{\Delta x}\left(\Delta x\, v_x - \frac{1}{2}(\Delta x)^2 v_{xx} - \dots\right) \end{aligned}$$

$\xrightarrow[\text{and re-arrange}]{\text{divide by } \Delta t}$

$$\Rightarrow \quad \left[v_t + \frac{1}{2}\Delta t\, v_{tt} + \frac{1}{6}(\Delta t)^2 v_{ttt} + \dots\right] + c\left[v_x - \frac{1}{2}\Delta x\, v_{xx} + \frac{1}{6}(\Delta x)^2 v_{xxx} - \dots\right] = 0$$

Modified equation [3]

re-write this as:

$$v_t + cv_x = \frac{1}{2}(c\Delta x v_{xx} - \Delta t v_{tt}) - \frac{1}{6}(c(\Delta x)^2 v_{xxx} + (\Delta t)^2 v_{ttt}) + \text{H.O.T.}$$

(higher order terms)

(this $\widetilde{\text{PDE}}$ is satisfied by $v(x,t)$)

4) let us neglect H.O.T.'s and define $\frac{\Delta t}{\Delta x} = \lambda$ (fixed)

$\Rightarrow$ the terms on the righthand side are $\mathcal{O}(\Delta t) + \mathcal{O}((\Delta t)^2)$

5) for small Δt, we can truncate the series to get a PDE that is quite well satisfied by u_i^n.
[note: dropping all terms on the righthand side gives the original advection PDE]

6) when we keep the $\mathcal{O}(\Delta t)$ terms, but neglect the $\mathcal{O}((\Delta t)^2)$ terms:

$$v_t + cv_x = \frac{1}{2}(c\Delta x v_{xx} - \Delta t v_{tt}) \qquad (**)$$

7) derive a slightly different equation:

a) differentiate w.r.t. t: $v_{tt} + cv_{xt} = \frac{1}{2}(c\Delta x v_{xxt} - \Delta t v_{ttt})$

b) differentiate w.r.t. x: $v_{tx} + cv_{xx} = \frac{1}{2}(c\Delta x v_{xxx} - \Delta t v_{ttx})$

8) Combine these two (using $v_{tx} = v_{xt}$):

$$v_{tt} = c^2 v_{xx} - \frac{c}{2}[c\Delta x v_{xxx} - \Delta t v_{ttx}] + \frac{1}{2}[c\Delta x v_{xxt} - \Delta t v_{ttt}]$$

Modified equation [4]

$$= c^2 v_{xx} + \mathcal{O}(\Delta t) \quad \text{with} \quad \frac{\Delta t}{\Delta x} = \lambda \text{ fixed}$$

9) insert 8) into $(**)$

$$\Rightarrow \quad v_t + c v_x = \frac{1}{2}(c\Delta x v_{xx} - c^2 \Delta t v_{xx}) + \mathcal{O}((\Delta t)^2)$$

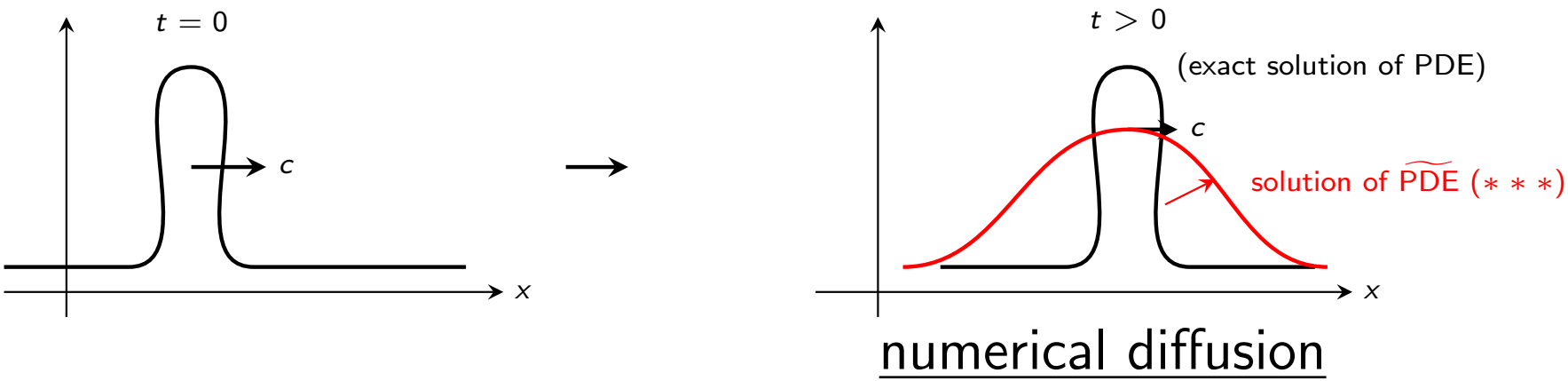
(drop this H.O.T. again)

$$\Rightarrow \quad \boxed{v_t + c v_x = \frac{1}{2} c\Delta x \left(1 - \frac{c\Delta t}{\Delta x}\right) v_{xx}} \qquad (***)$$

Conclusion: The u_i^n from the 1st order upwind for $u_t + c u_x = 0$ can be viewed as a 2nd order approximation of the exact solution of $(***)$.

- $\widetilde{\textbf{PDE}}$: $(***)$ is called the modified equation or modified PDE.
- Solutions of $(***)$ move with speed c, but are also diffused ("smeared out").

Modified equation [5]



Remark 1 (special case) if $c\Delta t = \Delta x$ then $\frac{1}{2}(c\Delta x - c^2\Delta t) = 0$
and $(***)$ becomes $v_t + cv_x = 0$

(in fact, this holds for all terms in the "H.O.T." expression that we have neglected!)

Remark 2 the diffusion coefficient $>$ 0 only if $0 < \dfrac{c\Delta t}{\Delta x} < 1$
numerical

("stable solution")

Modified equation [6]

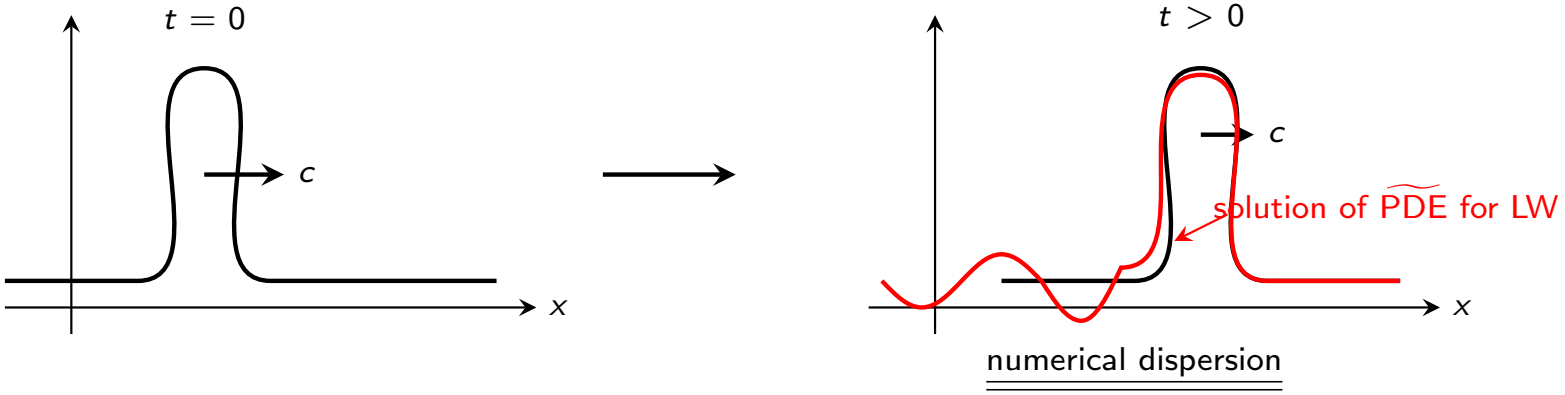
What about Lax-Wendroff?

Following the above recipe gives the modified PDE:

$$v_t + cv_x = -\frac{1}{6}c(\Delta x)^2\left(1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right)v_{xxx}$$

(third order derivative)

LW produces a 3^{rd} order approximation of this PDE



Modified equation [7]

$\Rightarrow$ "a train of oscillations" behind the solitary wave.

However, this "train" is limited, since (retaining one more term in recipe):

$$v_t + cv_x = -\frac{1}{6}c(\Delta x)^2\left(1-\left(\frac{c\Delta t}{\Delta x}\right)^2\right)v_{xxx} - \underbrace{\varepsilon}_{\substack{1\\ >0}} v_{xxxx}$$

Modified equation [8]

$\rightsquigarrow$ a higher-order "dissipation" (diffusion) causes the "train" to be damped (partially)
$\Rightarrow$ only a limited number of oscillations are seen in practice

(remember: $+u_{xx}$ damping, $-u_{xx}$ unstable, $+u_{xxxx}$ even more unstable, $-u_{xxxx}$ even more damping, etc...)

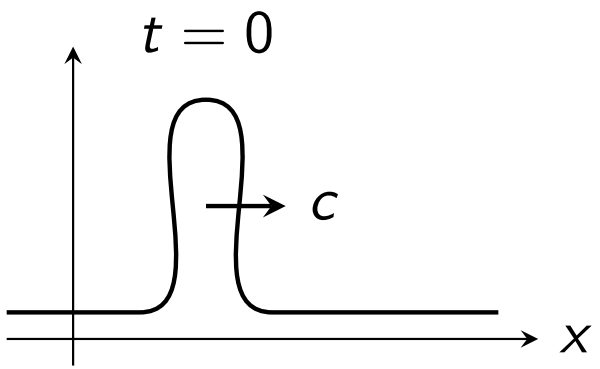
Modified equation [9]

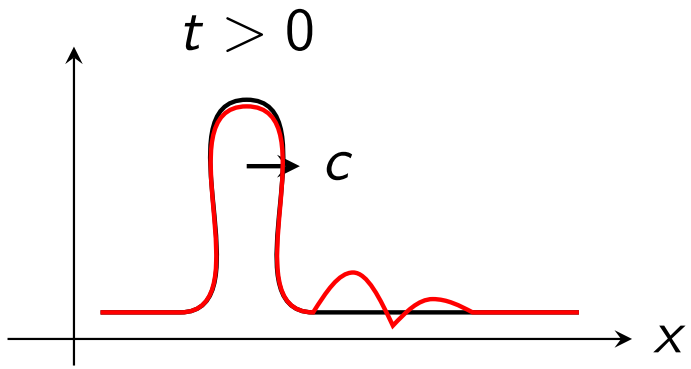
Beam-Warming method

$$\Longrightarrow \quad v_t + c v_x = \tfrac{1}{6} c(\Delta x)^2 \left(2 - \frac{3c\Delta t}{\Delta x} + \left(\frac{c\Delta t}{\Delta x}\right)^2\right) v_{xxx}$$

similar as for LW, but now the oscillations
move ahead of the wave, instead of behind, if $0 < \frac{c\Delta t}{\Delta x} < 1$

If $1 < \frac{c\Delta t}{\Delta x} < 2$, the oscillations are behind the wave, as in LW





$0 < \frac{c\Delta t}{\Delta x} < 1$

more on the CFL condition

Theorem Given "any" (consistent) three-point explicit FD scheme for $u_t + cu_x = 0$ of the form:

$$u_i^{n+1} = \alpha u_{i-1}^n + \beta u_i^n + \gamma u_{i+1}^n$$

Then a necessary (not sufficient!) condition for stability (and, therefore, for convergence) is the "CFL-condition":

$$\left| c \frac{\Delta t}{\Delta x} \right| \leq 1 \qquad \text{(Courant / Friedrichs / Lewy)}$$

Proof (see literature)

In other words: a necessary condition for stability is $\Delta t < \frac{\Delta x}{c}$.
Physically: time step Δt must be $<$ the time for a wave to travel across one grid interval
("time $= \frac{\text{distance}}{\text{speed}} = \frac{\Delta x}{c}$")

Note that, for example, Beam-Warming falls in another category of methods (CHECK)

Outlook to Lecture 6

- prepare exercises of Lecture 5 (see webpage)
- nonlinear hyperbolic PDEs
- wave equation
- more on the CFL-condition
- finite volumes