Numerical Methods for the 1D Wave Equation

Discretizations, CFL Stability Condition, and High-Order Schemes

Extra file 2 for Lecture 6

1. Introduction: The Continuous 1D Wave Equation

The 1D second-order linear hyperbolic partial differential equation is given by:

1D Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in (a, b),\ t > 0$$

where $u(x, t)$ is the displacement/wave amplitude and $c > 0$ is the wave propagation speed.

Initial Conditions (Cauchy Problem)

$$u(x, 0) = f(x) \quad \text{(Initial shape)}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) \quad \text{(Initial velocity)}$$

Physical Applications

- Vibrating strings & acoustic pressure waves.
- 1D electromagnetic waves.
- Elastic wave propagation in solids.

2. Analytical Solution: D'Alembert Formula

The wave equation can be factored into two counter-propagating advection equations:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0$$

D'Alembert's Exact Solution

$$u(x,t) = \phi(x - ct) + \psi(x + ct) = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

- **Characteristic Lines:** $x \pm ct = \text{constant}$.
- **Domain of Dependence:** The solution at (x_0, t_0) depends *only* on initial data within $[x_0 - ct_0,\, x_0 + ct_0]$.
- **Domain of Influence:** Localized initial disturbances propagate at exact speed c.

3. First-Order System Formulation (Hamiltonian / Flux Form)

The 2nd-order PDE can be converted into a system of two coupled 1st-order hyperbolic equations. Set $v = u_t$ (velocity) and $w = cu_x$ (proportional to strain):

1st-Order Vector System

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} v \\ w \end{pmatrix} \implies \mathbf{U}_t = \mathbf{A}\mathbf{U}_x$$

- Eigenvalues of matrix $\mathbf{A}$: $\lambda_{1,2} = \pm c$ (real and distinct $\rightarrow$ strictly hyperbolic).
- **Why use this?** Enables applying standard 1st-order CFD tools (e.g., Riemann solvers, MacCormack, Upwinding) directly to wave problems.

4. Method 1: Standard Explicit Finite Difference (CTCS / Leapfrog)

Discretize time ($t_n = n\Delta t$) and space ($x_j = j\Delta x$) using central differences:

Central-in-Time, Central-in-Space (CTCS) Scheme

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$$u_j^{n+1} = 2(1 - C^2)u_j^n + C^2\left(u_{j+1}^n + u_{j-1}^n\right) - u_j^{n-1}$$

where $C = \frac{c\Delta t}{\Delta x}$ is the **Courant number**.

- **Accuracy:** $\mathcal{O}(\Delta t^2 + \Delta x^2)$ (2nd-order in time and space).
- **Memory:** Requires two time levels (u^n and u^{n-1}).
- **First Step ($n = 0$):** Incorporate initial velocity $g(x)$ via ghost point $u_j^{-1} = u_j^1 - 2\Delta t\, g(x_j)$.

5. The Courant-Friedrichs-Lewy (CFL) Condition

CFL Condition (Necessary Condition for Convergence)

The numerical domain of dependence must contain the physical domain of dependence.

Mathematical Requirement

$$C = \frac{c\Delta t}{\Delta x} \leq 1 \implies \Delta t \leq \frac{\Delta x}{c}$$

Physical Interpretation

In a single time step Δt, physical waves cannot travel further than one spatial grid cell Δx.

- $C < 1$: Stable, but introduces numerical dispersion.
- $C = 1$: **Magic speed!** Shift operator becomes exact (zero numerical error for 1D grid).
- $C > 1$: Exponentially unstable!

6. Von Neumann Stability Analysis

Substitute harmonic mode $u_j^n = \xi^n e^{ikj\Delta x}$ into the CTCS scheme (where k is wavenumber):

Amplification Equation

$$\xi^2 - 2\left(1 - 2C^2 \sin^2\left(\frac{k\Delta x}{2}\right)\right)\xi + 1 = 0$$

Analysis Results:

- Product of roots: $\xi_1 \xi_2 = 1$.
- For stability, we require $|\xi_{1,2}| \leq 1 \implies |\xi_{1,2}| = 1$ (no dissipation, pure phase shift).
- Holds if and only if the discriminant is negative for all k:

$$-1 \leq 1 - 2C^2 \sin^2\left(\frac{k\Delta x}{2}\right) \leq 1 \implies C \leq 1$$

7. Method 2: Implicit θ-Scheme (Newmark / Crank-Nicolson)

Evaluate spatial derivatives at a weighted average of time levels $n-1$, n, and $n+1$:

General θ-Scheme

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \left[\theta \delta_x^2 u_j^{n+1} + (1-2\theta)\delta_x^2 u_j^n + \theta \delta_x^2 u_j^{n-1} \right]$$

where $\delta_x^2 u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}$.

Properties by Choice of Parameter θ:

- $\theta = 0$: Standard explicit CTCS scheme (CFL ≤ 1).
- $\theta = 1/4$: **Crank-Nicolson / Newmark-Beta** ($\beta = 1/4$). Unconditionally stable ($C < \infty$), 2nd-order in space and time.
- **Trade-off:** Requires solving a tridiagonal linear system at each time step!

8. Method 3: 4th-Order Spatial Finite Difference Scheme

To reduce spatial dispersion errors, replace the 3-point stencil with a 5-point central stencil:

4th-Order Space, 2nd-Order Time Scheme

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \left(\frac{-u_{j+2}^n + 16u_{j+1}^n - 30u_j^n + 16u_{j-1}^n - u_{j-2}^n}{12\Delta x^2} \right)$$

Basic Properties & Analysis:

- **Accuracy:** $\mathcal{O}(\Delta t^2 + \Delta x^4)$.
- **Strict CFL Condition:** Increased spatial stencil tightens the stability limit:

$$C = \frac{c\Delta t}{\Delta x} \leq \sqrt{\frac{3}{4}} \approx 0.866$$

- **Advantage:** Significantly reduces wave phase errors over long time simulations.

9. Method 4: High-Order Time Integrators (Method of Lines + RK4)

Convert PDE to a System of ODEs (Method of Lines): $\frac{d\mathbf{U}}{dt} = \mathbf{L}(\mathbf{U})$.

4th-Order Runge-Kutta (RK4) with 4th-Order Space

Discretize first-order system $\mathbf{U}_t = \mathbf{A}\mathbf{U}_x$ using 4th-order central spatial differences and integrate in time with 4th-order Runge-Kutta.

Basic Properties & Analysis:

- **Accuracy:** $\mathcal{O}(\Delta t^4 + \Delta x^4)$ (Fully 4th-order).
- **Properties:** High phase accuracy, minimal dissipation over thousands of wave periods.
- **CFL Limit:** Dependent on the stability region of RK4 along the imaginary axis ($C_{\max} \approx 1.24$ for 2nd-order space, ≈ 1.08 for 4th-order space).

10. Method 5: Pseudospectral Fourier Method

For periodic domains, represent $u(x,t)$ via discrete Fourier transforms (FFT):

Fourier Space ODEs

$$\frac{d^2\hat{u}_k(t)}{dt^2} = -c^2k^2\hat{u}_k(t) \quad \text{for wavenumber } k$$

Exact spatial derivative calculation in frequency domain: $\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -k^2\hat{u}_k$.

Basic Properties & Analysis:

- **Spatial Accuracy:** Spectral accuracy $\mathcal{O}(\Delta x^N)$ (exponential convergence for smooth solutions!).
- **Dispersion:** Zero spatial dispersion error for resolved wavenumbers.
- **CFL Requirement:** Extremely restrictive due to high maximum grid wavenumber $k_{\text{max}} = \pi/\Delta x$: $\Delta t \leq \frac{2}{\pi c}\Delta x \approx 0.637\frac{\Delta x}{c}$.

11. Numerical Error Analysis: Numerical Dispersion

Unlike physical waves where phase velocity $v_p = c$ is constant, numerical schemes introduce artificial **dispersion** ($v_p(\Delta x, \Delta t) \neq c$).

Numerical Phase Velocity (2nd-Order CTCS Scheme)

$$v_{p,\text{num}} = \frac{2}{k\Delta t} \arcsin\left(C \sin\left(\frac{k\Delta x}{2}\right)\right)$$

Effects of Dispersion

- High-frequency waves travel slower than physical speed ($v_p < c$).
- Causes spurious high-frequency "tail" oscillations behind wave pulses.

Remedies

- Increase grid resolution (points per wavelength $N_\lambda \geq 15 - 20$).
- Use higher-order schemes (4th-order FD or Spectral).
- Run at or near $C = 1$.

12. Comprehensive Scheme Comparison

Scheme	Time / Space Order	Type	CFL Limit (C)	Dispersion Error	Computational Cost
1. CTCS (Leapfrog)	2nd / 2nd	Explicit	≤ 1.000	Moderate	Low (3-point stencil)
2. Crank-Nicolson ($\theta = 1/4$)	2nd / 2nd	Implicit	Unconditional	Moderate	High (Tridiagonal matrix)
3. High-Order FD	2nd / 4th	Explicit	≤ 0.866	Low	Low-Medium (5-point)
4. MOL + RK4	4th / 4th	Explicit	≤ 1.080	Very Low	Medium (4 substeps)
5. Pseudospectral	Arbitrary / Spectral	Explicit	≤ 0.637	Zero (spatial)	Medium-High ($O(N \log N)$ FFT)

* Note: All explicit schemes require $C \leq C_{\max}$ to avoid catastrophic numerical instability.

13. Summary

1. **Respect the CFL Limit:** Explicit wave solvers *must* satisfy $C \leq C_{\max}$. Operating at $C > 1$ yields immediate divergence.
2. **Mind Grid Dispersion:** 2nd-order schemes require at least 15-20 grid points per wavelength (N_λ) to control phase lag over long distances.
3. **Optimal Setting for 2nd-order CTCS:** Setting $C = 1.0$ eliminates grid dispersion completely in 1D.
4. **For Long-Range Wave Propagation:** Prefer 4th-order (MOL+RK4) or Spectral methods to maintain phase accuracy and amplitude conservation.