The Method of Characteristics for Burgers' Equation

(Extra file 3 for Lecture 6)

1 Problem

We consider the nonlinear first-order partial differential equation

$$u_t + u\,u_x = 0, \qquad u(x,0) = u_0(x) = \sin(\pi x). \tag{1}$$

This is the inviscid Burgers' equation. The method of characteristics transforms the PDE into a system of ordinary differential equations along suitable curves in the $(x,t))$-plane.

The main questions are:

1. What are the characteristic curves?
2. How do we reconstruct $u(x,t)$ from them?
3. For how long does a classical solution exist?
4. What happens when the characteristics intersect?

2 General idea of the method of characteristics

Consider a first-order equation of the form

$$u_t + a(u)u_x = 0. \tag{2}$$

Suppose that $x = x(t)$ is a curve and define

$$U(t) = u(x(t), t).$$

By the chain rule,

$$\frac{dU}{dt} = u_t(x(t),t)) + x'(t)u_x(x(t),t). \tag{3}$$

If we choose

$$x'(t) = a(U(t)), \tag{4}$$

then the PDE gives

$$\frac{dU}{dt} = u_t + a(u)u_x = 0. \tag{5}$$

Thus U is constant along these curves.

For the Burgers' equation,

$$a(u) = u,$$

so the characteristic equations are

$$\boxed{\frac{dx}{dt} = u, \qquad \frac{du}{dt} = 0.} \tag{6}$$

3 Solving the characteristic equations

We parameterize the initial line $t = 0$ by a parameter ξ. At $t = 0$ we have

$$x(0) = \xi, \qquad u(0) = u_0(\xi) = \sin(\pi\xi). \tag{7}$$

Since

$$\frac{du}{dt} = 0,$$

the value of u remains constant along each characteristic:

$$u(t) = \sin(\pi\xi). \tag{8}$$

Consequently,

$$\frac{dx}{dt} = \sin(\pi\xi),$$

and integration gives

$$x(t) = \xi + t\sin(\pi\xi). \tag{9}$$

Hence the characteristics are described parametrically by

$$\boxed{x = \xi + t\sin(\pi\xi), \qquad u = \sin(\pi\xi).} \tag{10}$$

For every fixed ξ, this is a straight line in the (x, t)-plane:

$$x(t) = \xi + u_0(\xi)t. \tag{11}$$

Its slope is $u_0(\xi) = \sin(\pi\xi)$.

4 The solution in implicit form

The characteristic representation does not immediately give u explicitly as a function of (x, t), because ξ must first be determined from

$$x = \xi + t\sin(\pi\xi). \tag{12}$$

Once $\xi = \xi(x, t)$ has been found, the solution is

$$\boxed{u(x, t) = \sin(\pi\xi(x, t)).} \tag{13}$$

Equivalently, since

$$u = \sin(\pi\xi),$$

we have

$$\xi = x - tu.$$

Therefore the solution can also be written as the implicit equation

$$\boxed{u(x, t) = \sin\big(\pi(x - tu(x, t))\big).} \tag{14}$$

In general there is no convenient elementary formula for solving this last equation explicitly for u. The characteristic representation is therefore the natural exact description of the solution.

5 When does the classical solution break down?

The crucial issue is whether the map

$$\xi \longmapsto x(\xi, t) = \xi + t \sin(\pi\xi)$$

is invertible.

Different values of ξ must correspond to different positions x if we want a single-valued classical solution. We therefore compute

$$\frac{\partial x}{\partial \xi} = 1 + \pi t \cos(\pi\xi). \tag{15}$$

On the interval $0 \le \xi \le 1$, the smallest value of $\cos(\pi\xi)$ is -1, attained at $\xi = 1$. Thus the first time at which

$$x_\xi = 0$$

is determined by

$$1 - \pi t = 0.$$

Hence

$$\boxed{t_* = \frac{1}{\pi}.} \tag{16}$$

For

$$0 \le t < \frac{1}{\pi},$$

we have

$$1 + \pi t \cos(\pi\xi) > 0$$

for all $\xi \in [0, 1]$. Therefore the characteristic map is strictly increasing and the classical solution is single-valued.

The first singular point occurs at

$$\xi_* = 1, \qquad t_* = \frac{1}{\pi}.$$

Its spatial position is

$$x_* = 1 + \frac{1}{\pi}\sin(\pi) = 1. \tag{17}$$

Thus

$$\boxed{(x_*, t_*) = \left(1, \frac{1}{\pi}\right).} \tag{18}$$

6 Blow-up of the spatial derivative

The loss of invertibility can also be seen directly from u_x. We have

$$u(x, t) = u_0(\xi), \qquad x = \xi + t u_0(\xi).$$

Differentiating with respect to ξ gives

$$u_x x_\xi = u_0'(\xi).$$

Since

$$x_\xi = 1 + t u_0'(\xi),$$

we obtain the general formula

$$\boxed{u_x = \frac{u_0'(\xi)}{1 + t u_0'(\xi)}.} \tag{19}$$

For the present initial condition,

$$u_0'(\xi) = \pi \cos(\pi\xi),$$

and therefore

$$\boxed{u_x = \frac{\pi \cos(\pi\xi)}{1 + \pi t \cos(\pi\xi)}.} \tag{20}$$

At the point $\xi = 1$,

$$\cos(\pi) = -1,$$

so along the characteristic issuing from $x = 1$,

$$u_x(1, t) = -\frac{\pi}{1 - \pi t}. \tag{21}$$

Consequently,

$$u_x(t, 1) \longrightarrow -\infty \qquad \text{as} \qquad t \uparrow \frac{1}{\pi}. \tag{22}$$

Thus the solution itself remains bounded, but its spatial derivative becomes unbounded in finite time. This is the mechanism by which a shock develops in the inviscid Burgers' equation.

7 Geometric interpretation

The characteristic starting at ξ is

$$x(t) = \xi + t \sin(\pi\xi). \tag{23}$$

If $\sin(\pi\xi) > 0$, the characteristic moves to the right; if $\sin(\pi\xi) < 0$, it moves to the left.

On the interval $0 < \xi < 1$, all characteristic velocities are nonnegative. Near $\xi = 1$, however, the initial velocity decreases rapidly as ξ increases. Consequently, characteristics originating from points slightly to the left of $\xi = 1$ move faster than those originating closer to $\xi = 1$. The faster characteristics catch up with the slower ones.

At the critical time

$$t_* = \frac{1}{\pi},$$

the characteristic map first develops a vertical tangent when regarded as x as a function of ξ. After this time, different values of ξ can produce the same (x, t). The parametric characteristic solution is then multivalued.

This is why a classical solution cannot simply be continued beyond t_*.

8 General formula for the breaking time

For the inviscid Burgers' equation

$$u_t + u u_x = 0,$$

the characteristic solution is

$$x = \xi + t u_0(\xi), \qquad u = u_0(\xi).$$

The characteristic map loses invertibility when

$$x_\xi = 1 + t u_0'(\xi) = 0.$$

Therefore, if

$$\min_{\xi} u_0'(\xi) < 0,$$

the first breaking time is

$$\boxed{t_* = -\frac{1}{\min\limits_{\xi} u_0'(\xi)}.} \tag{24}$$

In our example,

$$u_0'(x) = \pi \cos(\pi x),$$

so

$$\min_{x} u_0'(x) = -\pi.$$

Hence

$$t_* = -\frac{1}{-\pi} = \frac{1}{\pi}.$$

9 Summary

For

$$u_t + u u_x = 0, \qquad u(0,x) = \sin(\pi x),$$

the method of characteristics gives

$$\boxed{x = \xi + t\sin(\pi\xi), \qquad u = \sin(\pi\xi).}$$

Before the breaking time,

$$0 \le t < \frac{1}{\pi},$$

the map $\xi \mapsto x$ is invertible and the unique classical solution is

$$\boxed{u(x,t) = \sin(\pi\xi(x,t)), \qquad x = \xi(x,t) + t\sin(\pi\xi(x,t)).}$$

The first singularity occurs at

$$\boxed{(x_*, t_*) = \left(1, \frac{1}{\pi}\right),}$$

where

$$u_x \to -\infty.$$

After this time the classical solution becomes multivalued. A physically relevant continuation is then obtained by interpreting Burgers' equation in the weak sense and selecting the entropy solution, which contains a shock.