Numerical and Theoretical Analysis of the Inviscid Burgers' Equation

Extra file 4 for Lecture 6

1 Continuous Theory of Non-Linear Hyperbolic Conservation Laws

1.1 Governing Formulations

The inviscid Burgers' equation is the simplest non-linear scalar PDE exhibiting wave steepening and shock formation.

Definition 1.1 (Advective / Non-Conservative Form)**.** Let $\Omega = \mathbb{R}\times(0,\infty)$. The non-conservative scalar advection equation is:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad u(x,0) = u_0(x). \tag{1}$$

Definition 1.2 (Conservative Flux Form)**.** Defining the convex flux function $f : \mathbb{R} \to \mathbb{R}$ by $f(u) = \frac{1}{2}u^2$, Eq. (1) becomes a one-dimensional conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \implies \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0. \tag{2}$$

1.2 Method of Characteristics and Wave Steepening

Consider a characteristic curve $x(t)$ in the space-time domain parameterized by initial position $x_0 \in \mathbb{R}$ such that $x(0) = x_0$.

Proposition 1.3 (Constancy along Characteristics)**.** *Along the parameter curve satisfying $\frac{dx}{dt} = u(x(t),t)$, the solution $u(x(t),t)$ is constant.*

Proof. By the multivariate chain rule:

$$\frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{3}$$

Thus $u(x(t),t) = u(x_0,0) = u_0(x_0)$. The characteristic curves are straight lines:

$$x(t) = x_0 + u_0(x_0)t. \tag{4}$$

□

1.3 Breaking Time and Shock Gradient Blow-Up

Differentiating the implicit solution $u(x,t) = u_0(x - u(x,t)t)$ with respect to x:

$$u_x(x,t) = u_0'(x_0)\left(1 - tu_x(x,t)\right) \implies u_x(x,t) = \frac{u_0'(x_0)}{1 + tu_0'(x_0)}. \tag{5}$$

Theorem 1.4 (Gradient Blow-Up / Breaking Time)**.** *If there exists any point $x_0 \in \mathbb{R}$ where $u_0'(x_0) < 0$, the spatial gradient $u_x(x,t)$ approaches $-\infty$ in finite time. The earliest breaking time t_b is given by:*

$$t_b = -\frac{1}{\min_{x_0 \in \mathbb{R}} u_0'(x_0)} > 0. \tag{6}$$

For $t \geq t_b$, characteristic lines intersect, rendering classical $\mathcal{C}^1$ solutions non-existent.

2 Weak Solutions, Jump Conditions, and Entropy Criteria

2.1 Weak Formulation

To accommodate discontinuous solutions for $t > t_b$, we generalize the PDE using test functions.

Definition 2.1 (Weak Solution)**.** A bounded measurable function $u \in L^\infty(\mathbb{R} \times [0,\infty))$ is a weak solution of Eq. (2) if for all compactly supported test functions $\phi \in \mathcal{C}_c^1(\mathbb{R} \times [0,\infty))$:

$$\int_0^\infty \int_{-\infty}^\infty \left(u \frac{\partial \phi}{\partial t} + f(u) \frac{\partial \phi}{\partial x} \right) dx\, dt + \int_{-\infty}^\infty u_0(x)\phi(x,0)\, dx = 0. \tag{7}$$

2.2 Rankine-Hugoniot Shock Jump Condition

Suppose a weak solution contains a smooth discontinuity curve $x = s(t)$ separating two continuous regions with states $u_L = \lim_{\epsilon \to 0^+} u(s(t) - \epsilon, t)$ and $u_R = \lim_{\epsilon \to 0^+} u(s(t) + \epsilon, t)$.

Theorem 2.2 (Rankine-Hugoniot Relation)**.** *The propagation speed $s' = \frac{dx_s}{dt}$ of a isolated shock wave satisfies:*

$$s' = \frac{[f(u)]}{[u]} = \frac{f(u_R) - f(u_L)}{u_R - u_L}. \tag{8}$$

For Burgers' flux $f(u) = \frac{1}{2}u^2$, the shock speed reduces to the arithmetic mean:

$$s' = \frac{\frac{1}{2}u_R^2 - \frac{1}{2}u_L^2}{u_R - u_L} = \frac{1}{2}(u_L + u_R). \tag{9}$$

2.3 Uniqueness Failure and Entropy Conditions

Weak solutions are non-unique. For example, a Riemann problem with $u_L = 0$ and $u_R = 1$ allows both an unstable jump discontinuity (an unphysical "expansion shock") and a continuous self-similar rarefaction wave.

Definition 2.3 (Lax Entropy Condition)**.** A discontinuity satisfying Rankine-Hugoniot is physically admissible (a shock rather than a rarefaction) if characteristics enter the shock line as time moves forward:

$$f'(u_L) > s' > f'(u_R) \implies u_L > s' > u_R. \tag{10}$$

For Burgers' equation, this requires $u_L > u_R$.

When $u_L < u_R$, the unique physical entropy solution is the smooth **rarefaction wave**:

$$u(x,t) = \begin{cases} u_L, & x < u_L t \\ x/t, & u_L t \leq x \leq u_R t \\ u_R, & x > u_R t. \end{cases} \tag{11}$$

3 Finite Volume Framework and the Lax-Wendroff Theorem

3.1 Conservative Finite Volume Discretization

We divide the domain into spatial cells $I_j = [x_{j-1/2}, x_{j+1/2}]$ of uniform width $\Delta x = x_{j+1/2} - x_{j-1/2}$, and time steps $\Delta t = t^{n+1} - t^n$.

Integrating Eq. (2) over $I_j \times [t^n, t^{n+1}]$ yields the exact conservative update equation for the cell average $u_j^n \approx \frac{1}{\Delta x}\int_{I_j} u(x, t^n)dx$:

$$u_j^{n+1} = u_j^n - \lambda\left(F_{j+1/2}^n - F_{j-1/2}^n\right), \quad \lambda = \frac{\Delta t}{\Delta x}, \tag{12}$$

where $F_{j+1/2}^n = \mathcal{F}(u_{j-k+1}^n, \ldots, u_{j+k}^n)$ is a consistent numerical flux approximating $\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{j+1/2}, t))dt$.

Definition 3.1 (Numerical Flux Consistency and Lipschitz Continuity)**.** A numerical flux $\mathcal{F}(u, v)$ is consistent if $\mathcal{F}(u, u) = f(u)$, and Lipschitz continuous if $|\mathcal{F}(u, v) - f(u)| \le L(|u - u| + |v - u|)$.

3.2 Failure of Non-Conservative Schemes (The Lax-Wendroff Theorem)

Theorem 3.2 (Lax-Wendroff Theorem (Lax & Wendroff, 1960))**.** *Let $u_{\Delta x}(x, t)$ be a sequence of numerical solutions generated by a* ***consistent and conservative*** *scheme (12). If $u_{\Delta x}$ converges boundedly almost everywhere to a limit function $u(x, t)$ as $\Delta x, \Delta t \to 0$, then $u(x, t)$ is a* ***weak solution*** *satisfying the Rankine-Hugoniot relations.*

Remark 3.3 (Consequence for Non-Conservative Schemes). If a scheme discretizes $u_t + uu_x = 0$ non-conservatively (e.g., $u_j^{n+1} = u_j^n - \lambda u_j^n(u_j^n - u_{j-1}^n)$), it violates local summation cancellation. As a result, even if the scheme converges, it converges to a function with the **wrong shock speed** $s_{\text{wrong}} \neq \frac{1}{2}(u_L + u_R)$, creating severe physical errors in CFD calculations.

4 Detailed Mathematical Analysis of Five Basic Schemes

In this section, we analyze five representative schemes for Burgers' equation. Let $a_j^n = u_j^n$ denote the local advection speed, and $\nu_j^n = u_j^n \frac{\Delta t}{\Delta x}$ denote the local Courant-Friedrichs-Lewy (CFL) number.

4.1 Scheme 1: Upwind Scheme (Non-Conservative)

The non-conservative First-Order Upwind scheme applies a directional spatial difference:

$$u_j^{n+1} = u_j^n - \lambda\left[\max(u_j^n, 0)(u_j^n - u_{j-1}^n) + \min(u_j^n, 0)(u_{j+1}^n - u_j^n)\right]. \tag{13}$$

Mathematical Properties

- **Formulation:** Non-Conservative Finite Difference.
- **Truncation Error Analysis:** Assuming $u > 0$, Taylor expansions yield:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + u_j^n \frac{u_j^n - u_{j-1}^n}{\Delta x} = u_t + uu_x - \frac{\Delta x}{2} u\left(1 - \frac{u\Delta t}{\Delta x}\right) u_{xx} + \mathcal{O}(\Delta t^2, \Delta x^2). \tag{14}$$

The term $d_{\text{num}} = \frac{u\Delta x}{2}(1 - \nu)$ acts as explicit **numerical diffusion**.

- **Stability Analysis:** Linearizing around a constant state $U > 0$ gives $\nu = U\frac{\Delta t}{\Delta x}$. The amplification factor $G(\theta)$ for Fourier mode $e^{ij\theta}$ ($\theta = k\Delta x$) is:

$$G(\theta) = 1 - \nu(1 - e^{-i\theta}) = (1 - \nu + \nu\cos\theta) - i\nu\sin\theta. \tag{15}$$

$$|G(\theta)|^2 = 1 - 4\nu(1-\nu)\sin^2(\theta/2) \leq 1 \iff 0 \leq \nu \leq 1. \tag{16}$$

- **Shock Performance: Fails Lax-Wendroff test.** Converges to incorrect shock speeds.

4.2 Scheme 2: Lax-Friedrichs Scheme (Conservative)

Replaces the central node u_j^n in an unstable FTCS scheme with the spatial average $\frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$:

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\lambda}{2}\left(\frac{1}{2}(u_{j+1}^n)^2 - \frac{1}{2}(u_{j-1}^n)^2\right). \tag{17}$$

The conservative interface flux is:

$$F_{j+1/2}^{\text{LF}} = \frac{1}{2}\left(f(u_j^n) + f(u_{j+1}^n)\right) - \frac{1}{2\lambda}(u_{j+1}^n - u_j^n). \tag{18}$$

Mathematical Properties

- **Formulation:** Conservative Finite Volume.
- **Modified Equation:**

$$u_t + f(u)_x = \left(\frac{\Delta x^2}{2\Delta t} - \frac{\Delta t}{2}f'(u)^2\right)u_{xx} = \frac{\Delta x^2}{2\Delta t}(1-\nu^2)u_{xx}. \tag{19}$$

  The numerical viscosity coefficient $\frac{\Delta x^2}{2\Delta t}$ is large, proportional to $\mathcal{O}(\Delta x^2/\Delta t)$.

- **Monotonicity:** The scheme is a **monotone scheme** (and hence Total Variation Diminishing) provided the CFL condition $\max|u|\frac{\Delta t}{\Delta x} \leq 1$ holds.
- **Shock Performance:** Converges to the correct entropy solution, but severely smears sharp shock profiles due to high artificial viscosity.

4.3 Scheme 3: Lax-Wendroff Scheme (Conservative)

Derived via a second-order Cauchy-Kowalevski Taylor expansion in time ($u_{tt} = (-(f(u))_x)_t = (f'(u)f(u)_x)_x$):

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{\lambda^2}{2}\left[A_{j+1/2}^n(f_{j+1}^n - f_j^n) - A_{j-1/2}^n(f_j^n - f_{j-1}^n)\right], \tag{20}$$

where $A(u) = f'(u) = u$, and $A_{j+1/2}^n = \frac{1}{2}(u_j^n + u_{j+1}^n)$. The conservative interface flux is:

$$F_{j+1/2}^{\text{LW}} = \frac{1}{2}(f_j + f_{j+1}) - \frac{\lambda}{2}A_{j+1/2}(f_{j+1} - f_j). \tag{21}$$

Mathematical Properties

- **Accuracy:** Second-order in space and time $\mathcal{O}(\Delta t^2, \Delta x^2)$.
- **Modified Equation Dispersion:**

$$u_t + f(u)_x = -\frac{\Delta x^2}{6} u \left(1 - \nu^2\right) u_{xxx} + \mathcal{O}(\Delta x^4). \tag{22}$$

  The leading error term is third-order spatial derivative (u_{xxx}), which causes **dispersion waves**.
- **Godunov's Barrier Theorem:** Godunov (1959) proved that no linear scheme of order higher than one can be non-oscillatory. Consequently, Lax-Wendroff exhibits unphysical **spurious oscillations** (overshoots and undershoots) near shock discontinuities.

4.4 Scheme 4: MacCormack Scheme (Predictor-Corrector)

A two-step predictor-corrector Finite Difference scheme that avoids calculating explicit Jacobians:

$$\textbf{Predictor (Forward):} \quad \bar{u}_j^{n+1} = u_j^n - \lambda u_j^n \left(u_{j+1}^n - u_j^n\right), \tag{23}$$

$$\textbf{Corrector (Backward):} \quad u_j^{n+1} = \frac{1}{2}\left[u_j^n + \bar{u}_j^{n+1} - \lambda \bar{u}_j^{n+1}\left(\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}\right)\right]. \tag{24}$$

Mathematical Properties

- **Accuracy:** Second-order accurate $\mathcal{O}(\Delta t^2, \Delta x^2)$ for smooth flows. Identical to Lax-Wendroff for linear equations.
- **Conservation Status:** As written above using advective terms uu_x, it is **non-conservative**. However, if implemented on flux vectors $f(u)$ ($\bar{u}_j = u_j - \lambda(f_{j+1} - f_j)$), it becomes fully conservative.
- **Stability:** Stable for CFL ≤ 1. Produces post-shock numerical oscillations similar to Lax-Wendroff.

4.5 Scheme 5: Engquist-Osher E-Flux Scheme (Conservative)

An advanced first-order upwind flux-splitting method designed to satisfy entropy conditions strictly without expansion shocks. The Engquist-Osher numerical flux is defined by integral splitting:

$$F_{j+1/2}^{\mathrm{EO}} = \frac{1}{2}\left(f(u_j) + f(u_{j+1})\right) - \frac{1}{2}\int_{u_j}^{u_{j+1}} |f'(u)|\, du. \tag{25}$$

For $f(u) = \frac{1}{2}u^2$, where $f'(u) = u$, this simplifies to:

$$F_{j+1/2}^{\mathrm{EO}} = f^+(u_j^n) + f^-(u_{j+1}^n), \tag{26}$$

where $f^+(u) = \frac{1}{2}(\max(u, 0))^2$ and $f^-(u) = \frac{1}{2}(\min(u, 0))^2$.

Mathematical Properties

- **Formulation:** Conservative, Monotone, and TVD Finite Volume Scheme.
- **E-Flux Condition (Osher, 1980):** The numerical flux satisfies $\mathrm{sgn}(u_{j+1} - u_j)(F_{j+1/2}^{\mathrm{EO}} - f(u)) \leq 0$ for all u between u_j and u_{j+1}.
- **Entropy Satisfaction:** Automatically eliminates sonic non-physical expansion shocks at $u = 0$ (transonic rarefactions) without needing an explicit entropy fix.

5 High-Resolution TVD Schemes and Flux Limiters

To resolve the conflict between Godunov's Theorem (1st-order schemes are non-oscillatory but diffusive; 2nd-order schemes are non-diffusive but oscillatory), modern CFD uses non-linear High-Resolution TVD schemes.

Definition 5.1 (Total Variation)**.** The Total Variation of a discrete numerical solution u^n is defined as:

$$TV(u^n) = \sum_{j=-\infty}^{\infty} |u_{j+1}^n - u_j^n|. \tag{27}$$

A scheme is **Total Variation Diminishing (TVD)** if $TV(u^{n+1}) \leq TV(u^n)$ for all n.

5.1 Flux Limiter Formulation

We construct a high-resolution TVD flux by blending a low-order diffusive flux F^{low} (e.g., Upwind) with a high-order non-diffusive flux F^{high} (e.g., Lax-Wendroff) using a non-linear flux limiter function $\phi(r)$:

$$F_{j+1/2}^{\text{TVD}} = F_{j+1/2}^{\text{low}} + \phi(r_j)\left(F_{j+1/2}^{\text{high}} - F_{j+1/2}^{\text{low}}\right), \tag{28}$$

where r_j monitors local solution smoothness (ratio of successive gradients):

$$r_j = \frac{u_j^n - u_{j-1}^n}{u_{j+1}^n - u_j^n}. \tag{29}$$

Theorem 5.2 (Sweby's TVD Region (Sweby, 1984))**.** *A scheme is TVD under CFL* ≤ 1 *if the limiter function* $\phi(r)$ *falls within the Sweby TVD region bounded by* $\phi(r) = 0$ *for* $r \leq 0$*, and:*

$$0 \leq \left(\frac{\phi(r)}{r}, \phi(r)\right) \leq 2. \tag{30}$$

Common second-order limiter functions include:

- **Minmod Limiter:** $\phi_{\text{MM}}(r) = \max(0, \min(1, r))$.
- **van Leer Limiter:** $\phi_{\text{VL}}(r) = \frac{r+|r|}{1+|r|}$.
- **Superbee Limiter:** $\phi_{\text{SB}}(r) = \max(0, \min(2r, 1), \min(r, 2))$.

6 Comparative Numerical Analysis Summary

Numerical Scheme	**Formulation Type**	**Formal Order**	**CFL Condition**	**Lax-Wendroff Compliance**	**Shock & Oscillatory Behavior**
Upwind	Non-Conservative	$\mathcal{O}(\Delta x)$	≤ 1	**Violated**	Incorrect shock speed; non-oscillatory.
Lax-Friedrichs	Conservative	$\mathcal{O}(\Delta x)$	≤ 1	Satisfied	Correct shock speed; excessively diffusive.
Lax-Wendroff	Conservative	$\mathcal{O}(\Delta x^2)$	≤ 1	Satisfied	Correct shock speed; severe overshoot oscillations.
MacCormack	Non-Cons.* / Cons.	$\mathcal{O}(\Delta x^2)$	≤ 1	Conditional*	Oscillatory near steep gradients.
Engquist-Osher	Conservative	$\mathcal{O}(\Delta x)$	≤ 1	Satisfied	Correct shock speed; TVD non-oscillatory.
TVD Limiters	Conservative	$\mathcal{O}(\Delta x^2)$	≤ 1	Satisfied	Sharp non-oscillatory shock capture.

Table 1: Comprehensive comparison of numerical schemes for the inviscid Burgers' equation $u_t + uu_x = 0$.