DIFFERENTIAL EQUATIONS
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DIFFERENCE EQUATION MODELS OF DIFFERENTIAL EQUATIONS HAVING ZERO LOCAL TRUNCATION ERRORS

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It is suggested that there exists a "best" difference equation model of a differential equation. An existence theorem is proven for the case of ordinary differential equations. The difficulties of generalizing this result to partial differential equations are discussed.

INTRODUCTION

Most differential equations which occur in scientific applications are nonlinear and exact analytic solutions either are not known or, if known, can not be put in a form suitable for useful calculations (Fornberg and Whitham, 1978; Hyman, 1982; Potter, 1973; Whitham, 1974). The situation is hardly better for linear differential equations where the coefficients are functions of the independent variable(s). In practice, this situation leads one to attempt a numerical integration of the equation(s) of interest (Carrier and Pearson, 1976; Hildebrand, 1968). Of the many numerical techniques, the method of finite differences is of particular importance (Mitchell, 1969) since their solutions can, in general, be readily obtained by use of high speed computers.

The question immediately arises as to whether there is a "best" difference equation model of a differential equation. In the next section, a discussion of such schemes is presented for ordinary differential equations (ODE's). For our purposes, the "best" difference scheme turns out to be one which has zero local truncation errors. There are a number of reasons why we would seek such schemes; they include (i) the fact that the step sizes can be chosen to have an arbitrary value (for a particular problem the appropriate step size would be determined by physical considerations) and (ii) questions of convergence, stability and consistency (Hildebrand, 1968) would not arise.

In the third section, we attempt to generalize the results obtained for ODE's to partial differential equations (PDE's). Some fundamental difficulties arise. Finally, in the last section, related areas of investigation are discussed.

ORDINARY DIFFERENTIAL EQUATIONS

Consider a first-order ODE

$$\frac{dx}{dt} = f(t,x) \quad , \quad x(t_o) = x_o \tag{1}$$

and a first-order ordinary difference equation (OΔE)

$$x_{k+1} = g(k,x_k) \ . \tag{2}$$

Consider the following question (Potts, 1982; Van der Vaart, 1973): Given an arbitrary first-order ODE, can a first-order OΔE be determined that has the same general solution? By the same general solution, we mean, for an arbitrary

constant step-size h, the values x_k satisfying Eq. (2) are related to the solution x(t) of Eq. (1) by

(3) $$x(hk) = x_k \quad \text{where} \quad t \to t_k = hk.$$

From a geometrical viewpoint this means that x_k satisfying Eq. (2) will be points on the solution curve x(t) of Eq. (1) at successive values t = hk, k an integer. If the above question can be answered in the affirmative, then it means that it is possible to construct a finite difference model of an ODE that has zero local truncation error. Such a scheme we can identify as being the "best" infinite difference model of the original ODE. We now show that if Eq. (1) has a well-defined solution, then a finite difference scheme having this property exists.

Assume that the general solution to Eq. (1) is known and is given by

(3a) $$x(t) = \phi(t, t_o, x_o) \quad ,$$

where x_o is the value of x(t) at $t = t_o$. The "group property" (Nemytski and Stepanov, 1969) of the solution allows the result

(4) $$x(t + h) = \phi(t + h, t, x(t))$$

where h is an arbitrary constant. If we now take t = kh, x(hk) = x_k, then Eq. (4) becomes

(5) $$x_{k+1} = \phi(h(k+1), hk, x_k)$$

and this is the explicit form of the required OΔE which has the same general solution as Eq. (1). This method can be easily generalized to determine a system of nonlinear OΔE's that has the same general solution as a system of nonlinear ODE's.

The following two elementary examples illustrate the above technique.

	ODE	OΔE
(6)	$\frac{d^2x}{dt^2} + w^2 x = 0$	$x_{k+1} - 2x_k + x_{k-1} + 4x_k \sin^2(\frac{wh}{2}) = 0$
(7)	$\frac{dx}{dt} + x = \frac{A}{x}$	$x_{k+1}^2 = (x_k^2 - A)\exp(-2h) + A$

From Eq. (4) it is seen that the above procedure leads to an explicit finite difference model for the corresponding differential equation. Further, it should be clear that all we have is essentially an existence theorem which allows us to conclude that such schemes exist, but not how to construct them for an arbitrary ODE. In general, there are two problems which arise in the construction of finite difference models of ODE's. They relate directly to how one should model the, respective, derivative and nonlinear terms of the ODE by finite difference expressions. First, the replacement of a derivative term of a given order by a finite difference expression of a higher order will, in general, lead to "chaotic" behavior in the solution to the OΔE which is not present in the ODE (Hildebrand, 1968; Yamaguti and Ushiki, 1981). Our conclusion is that in the construction of the "best" OΔE corresponding to an ODE, the order of the OΔE and ODE must be the same. Second, there is a nonuniqueness as to how the nonlinear terms in the ODE should be modeled. For example, in the equation

(8) $$\frac{d^2x}{dt^2} + x^3 = 0 \quad ,$$

$x(t) = C_1 \cos(wt) + C_2 \sin(wt)$

the nonlinear term could, in principle, be represented by any one of the following expressions.

(9) x_k^3 , $x_{k+1}(x_k^2 + x_{k-1}^2)/2$, $(x_{k+1}^3 + x_k^3 + x_{k-1}^3)/3$,

none of which when combined with standard representations for the second-order derivative term will give the "best" difference scheme as defined above.

At the present time, the way to proceed is probably through the experimental construction of difference schemes for particular classes of ODE's in the hope that eventually a set of rules will emerge for obtaining the "best" OΔE models for members of these classes. Given an OΔE model of a ODE, we must check it to see if it is the "best" model. This can be done by calculating the local truncation error; if it is zero, then the scheme is the "best".

PARTIAL DIFFERENTIAL EQUATIONS

A major complication in trying to construct finite-difference models of PDE's is the lack of an unambiguous definition of a general solution to a PDE. This is, in part, reflected in the fact that one can formally derive finite-difference schemes which have zero local truncation errors but are of infinite order. For example, the parabolic equation for $u(x,t)$

(10) $u_t = L(t,x,D,D^2)u$

where L is a linear operator and $D = \partial/\partial x$ has the following "best" finite-difference model (Mitchell, 1969)

(11) $u_m^{n+1} = \text{expkL}(hm, kn, \frac{2}{h}\sinh^{-1}\frac{\delta x}{2}, (\frac{2}{h}\sinh^{-1}\frac{\delta x}{2})^2)\, u_m^n$

$\delta_x = \dfrac{u_{m+1}^n - u_{m-1}^n}{2h}$

where the x and t grid spacings are h and k, and δ_x is the central difference operator in the x-direction (Hildebrand, 1968). Note that while this is an explicit two-level time scheme, it is formally of infinite order in the discrete space variable. However, there are a number of PDE's for which the "best" difference scheme can be found; they include the advective and wave equations:

ODE PDE		OΔE PΔE	
(12)	$u_t + u_x = 0$	$u_m^{n+1} = u_{m-1}^n$	
(13)	$u_{tt} - u_{xx} = 0$	$u_m^{n+1} = u_{m+1}^n - u_m^{n-1} + u_{m-1}^n$	

$\Delta t = \Delta x$





where in each case $\Delta x = \Delta t$. Again, as in the case for ODE's, we must for PDE's "experiment" with particular classes of PDE's in search of possible rules for constructing the "best" finite-difference models. Recently, we have begun such a study on the two equations

(14) $u_t + uu_x = 0$ $\qquad$ $u_t = auu_{xx}$

where a is a constant. Interesting and positive results have been obtained.

DISCUSSION

The above comments readily show that the general task of constructing "best" finite-difference models of ODE's and PDE's is difficult. However, the difficulty is lessened somewhat by the realization that in the analysis of physical problems the differential equations of interest are generally limited to a rather small class of equations (Potter, 1973). This fact at least shows us where the initial efforts should be placed. Even partial success could give important results. For example, in the construction of finite-difference schemes for nonlinear ODE's with constant coefficients, if the nonlinear term can be considered a small

perturbation, then a useful (though not the "best") model might consist of replacing the linear part of the ODE by its "best" finite-difference counterpart and then modeling the nonlinear part in some appropriate fashion. Thus for one-dimensional nonlinear oscillators one can insure that the unperturbed period has the correct value; standard methods give finite-difference equations which do not have this property (Potts, 1982).

In view of these last remarks, it would be of importance to have perturbation techniques available for the construction of approximate analytic solutions to the difference equations. To date, these efforts have not been very productive since the solutions contain secular terms (Huston, 1970; Torng, 1960). Recently, we have begun a detailed study of series solutions to a particular class of nonlinear difference equations having small nonlinearities. It was shown that uniformly valid asymptotic series solutions could be obtained (Mickens, 1982). Efforts to extend these calculations to partial difference equations are presently underway.

Finally, it should be mentioned that a number of researchers have invented methods for constructing finite-difference analogues of nonlinear PDE's which exhibit either special types of solutions (e.g., solitons) or exact solutions. The recent paper of Hirota (1982) provides a good summary of the results obtained to date.

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REFERENCES

(1) Carrier, G. F. and Pearson, C. E., Partial Differential Equations: Theory and Technique (Academic, New York, 1976).

(2) Fornberg, B. and Whitham, G. B., A Numerical and Theoretical Study of Certain Nonlinear Wave Phenomena, Phil. Trans. of Royal Soc. (London) 289 (1978) 373-404.

(3) Greenspan, D., Discrete Models (Addison-Wesley; Reading, MA; 1973).

(4) Hildebrand, F. B., Finite-Difference Equations and Simulations (Prentice-Hall; Englewood Cliffs, NJ; 1968).

(5) Hirota, R., Difference Analogues of Nonlinear Evolution Equations in Hamiltonian Form, Tech. Rep. No. A-12, Hiroshima University (August 1982).

(6) Huston, R. L., Krylov-Bogolijubov Method for Difference Equations, SIAM J. Appl. Math. 19 (1970) 334-339.

(7) Hyman, J. M., Numerical Methods for Nonlinear Differential Equations, LA-8927-MS, Los Alamos National Laboratory (January 1982).

(8) Mickens, R. E., A Method for Obtaining Periodic Solutions to A Particular Class of Nonlinear Difference Equations, J. of Franklin Institute (accepted for publication, 1982).

(9) Mitchell, A. R., Computational Methods in Partial Differential Equations (Wiley, New York, 1969).

(10) Nemytski, V. V. and Stepanov, V. V., Qualitative Theory of Differential Equations (Princeton University Press; Princeton, NJ; 1969).

(11) Potter, D., Computational Physics (Wiley, New York, 1973).

(12) Potts, R. B., Differential and Difference Equations, Am. Math. Monthly 89, (1982) 402-407.

(13) Torng, H. C., Second-Order Nonlinear Difference Equations Containing Small Parameters, J. of Franklin Institute 269 (1960) 97-104.

(14) Van der Vaart, A Comparative Investigation of Certain Difference Equations and Related Differential Equations: Implications for Model Building, Bull. Math. Biology 35 (1973) 195-211.

(15) Whitham, G. B., Linear and Nonlinear Waves (Wiley-Interscience, New York, 1974).

(16) Yamaguti, M. and Ushiki, S., Chaos in Numerical Analysis of Ordinary Differential Equations, Physica 3D (1981) 618-626.