Dispersive Waves in $u_t = \pm u_{xxx}$: Why Oscillations Appear on One Side of the Wave

1 The Intuitive Picture: Waves Traveling at Different Speeds

Consider the initial value problem on $\mathbb{R}$ for $t > 0$:

$$u_t = \pm u_{xxx}, \qquad u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{1}$$

This is the simplest example of a **linear dispersive partial differential equation** (also known as the linearized Korteweg–de Vries equation or the Airy equation).

To understand in the simplest terms why an oscillatory tail appears on one side of the main wave—and why switching the $+$ and $-$ signs flips which side it appears on—we break the solution down into three intuitive steps:

1. **Any initial bump is a sum of simple sine waves:**
   Imagine $u_0(x)$ as a single localized bump (such as a Gaussian pulse). By Fourier analysis, this bump is built by adding together infinitely many pure sine and cosine waves of different wavelengths:

   - *Long, broad waves* with small wave number $k \approx 0$.
   - *Short, rapid ripples* with large wave number $|k| \gg 1$.

   At $t = 0$, all these waves line up constructively to form the single bump and cancel each other out everywhere else.

2. **Wave speed depends on wavelength (Dispersion):**
   Substitute a single plane wave $u(x,t) = e^{i(kx-\omega t)}$ into the PDE:

   - For $u_t = -u_{xxx}$, we get $-i\omega = -(ik)^3 = ik^3 \implies \omega = -k^3$.
     The phase velocity (speed of the wave crests) is

     $$c_{\text{phase}} = \frac{\omega}{k} = -k^2 \quad \text{(wait: in } e^{i(kx-\omega t)}\text{, } \omega = -k^3 \text{ means } e^{i(kx+k^3t)} = e^{ik(x+k^2t)}\text{, moving left!)}.$$

     Sign convention check: Let us be completely explicit by writing the wave as $e^{ik(x-vt)}$, where v is the velocity toward $+x$:

     $$u_t = -u_{xxx} \implies -ikv = -(ik)^3 = ik^3 \implies v = -k^2 \quad \text{(moves **left**, toward } -\infty\text{)},$$
     $$u_t = +u_{xxx} \implies -ikv = +(ik)^3 = -ik^3 \implies v = +k^2 \quad \text{(moves **right**, toward } +\infty\text{)}.$$

     (Note: Equivalently, for the standard KdV convention $u_t + u_{xxx} = 0$, i.e., $u_t = -u_{xxx}$, waves travel to the left; for $u_t - u_{xxx} = 0$, i.e., $u_t = +u_{xxx}$, waves travel to the right.)

   The crucial point: In both cases, the speed $|v| = k^2$ grows with k^2. **Short ripples (large $|k|$) travel much faster than long waves (small $|k|$).**

3. **Why oscillations appear on only one side:**
   As soon as $t > 0$, the waves separate because they travel at different speeds:

- **Case 1: $u_t = +u_{xxx}$ (velocity $v = +k^2 > 0$, moving right):**
  Short, high-frequency ripples ($|k|$ large) race far ahead to the **right** ($x > 0$). Long, broad waves ($|k|$ small) barely move and stay near $x \approx 0$. Result: a smooth, rapidly decaying front on the left ($x < 0$) and a long train of oscillations spreading out to the **right** ($x > 0$).
- **Case 2: $u_t = -u_{xxx}$ (velocity $v = -k^2 < 0$, moving left):**
  Everything is mirror-imaged. All waves move left, and the short ripples race far ahead to the **left** ($x < 0$). Result: rapid oscillations spread out to the **left** ($x < 0$), while the right side ($x > 0$) decays smoothly without oscillations.

Analogy: Think of runners lined up at $x = 0$ at $t = 0$. The heavy, slow runners (long waves) stay near the starting line, while the light, fast sprinters (short ripples) run ahead. The $+$ or $-$ sign in $u_t = \pm u_{xxx}$ simply decides whether the race track goes to the right or to the left: the fast ripples always lead the pack!