Computer Exercise C1A

Numerical Methods for PDEs

5% of Final Grade

Deadline: See the course webpage.

Individual Work: This is an individual assignment.

Submission: Send your report to: `numpde2026ATgmail.com`.

PART A (45 points)

We are looking for a solution of the *backward* wave equation:

$$u_{tt} = -u_{xx},$$

with initial conditions $u(x,0) = \sin(2\pi x)$ and $u_t(x,0) = 0$, and with periodic boundary conditions $u(0,t) = u(1,t)$ and $u_x(0,t) = u_x(1,t)$.

The solution to this equation can be written in the form of a Fourier series:

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\sin(2\pi kx) + \sum_{k=0}^{\infty} b_k(t)\cos(2\pi kx).$$

(a) Show that $a_k(t) = b_k(t) = 0$ for all $k \neq 1$ and that:

$$\begin{cases} \ddot{a}_1(t) - 4\pi^2 a_1(t) = 0, \\ \ddot{b}_1(t) - 4\pi^2 b_1(t) = 0. \end{cases}$$

with $a_1(0) = 1$ and $a_1'(0) = b_1(0) = b_1'(0) = 0$.

(b) Conclude that the solution can be written as $u(x,t) = \cosh(2\pi t)\sin(2\pi x)$.

PART B (55 points)

Consider the one-dimensional *stationary* convection-diffusion model:

$$\begin{cases} -\epsilon u''(x) + u'(x) = 0, \ x \in [0,1], \ \epsilon > 0, \\ u(0) = 0, \quad u(1) = 1. \end{cases} \tag{1}$$

The exact solution reads: $u(x) = \left(\exp\left(\frac{x}{\epsilon}\right) - 1\right) / \left(\exp\left(\frac{1}{\epsilon}\right) - 1\right)$.

(a) Work out: 1) a central difference scheme (both for u' and u'') and 2) a one-sided difference[1] scheme (for u', but still central for u'') in equation (1). Write the fully-discretized $(I-1)\times(I-1)$ linear system as: $\mathcal{A}\vec{u} = \vec{f}$, where

$$\mathcal{A} = \frac{1}{(\Delta x)^2} \begin{pmatrix} b & c & 0 & & & & \\ a & b & c & 0 & & \emptyset & \\ 0 & a & b & c & 0 & & \\ & & \ldots & \ldots & \ldots & & \\ & & & \ldots & \ldots & \ldots & \\ & & & 0 & a & b & c & 0 \\ & \emptyset & & & 0 & a & b & c \\ & & & & & 0 & a & b \end{pmatrix},$$

and

$$\vec{u} = [u_1, u_2, ..., u_{I-2}, u_{I-1}]^T, \ \vec{f} = [f_1, 0, ..., 0, f_{I-1}]^T.$$

Find the values a, b, c, f_1 and f_{I-1} for both schemes.

(b) Solve the following linear difference equations:

$$a\ u_{i-1} + b\ u_i + c\ u_{i+1} = 0, \quad i = 1, ..., I-1,$$

$$\text{where } a < 0, \ b > 0, \ a + b + c = 0, \ \tfrac{a}{c} \neq \pm 1, \ c \neq 0$$

by using a trial solution of the form: $u_i = \alpha + \beta r^i$ with $u_0 = 0, \ u_I = 1$.

(c) Consider the two cases $c > 0$ and $c < 0$. Check for the solutions in part (b), whether we can expect *positivity-preserving* numerical solutions ($u_i \geq 0$) and/or whether we can expect solutions with *non-physical oscillations* in the numerical values u_i. Show that neither of the difference schemes of (a) produces *oscillations* when $\Delta x < 2\epsilon$ holds.

(d) Next, consider the reaction-diffusion model:

$$\begin{cases} -\epsilon u''(x) + u(x) = 0, \ x \in [0,1], \ \epsilon > 0, \\ u(0) = 0, \quad u(1) = 1. \end{cases} \tag{2}$$

Use a trial function of the form:

$$u_i = \left(\tau_2^i - \tau_1^i\right) / \left(\tau_2^I - \tau_1^I\right), \quad \forall i. \tag{3}$$

Find the constants τ_1 and τ_2 for the central difference approximation of (2), and show that the scheme yields solutions without numerical oscillations.

[1] Approximation: $u'|_{x_i} \approx \frac{u_i - u_{i-1}}{\Delta x}$.