Computer Exercise C1B

Numerical Methods for PDEs

10% of Final Grade

Deadline: See the course webpage.

Individual Work: This is an individual assignment.

Submission: Send your report and all MATLAB (or Python) codes to: `numpde2026ATgmail.com`.

PART A (50 points)

Consider the one-dimensional advection equation:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad x \in (0,1),\ t \in (0,T), \tag{1}$$

with the initial condition:

$$u(x,0) = u_0(x) = \sin(2\pi x) + \frac{1}{2}\sin(4\pi x).$$

The equation is subject to periodic boundary conditions:

$$u(0,t) = u(1,t).$$

(a) Show that the following quantity is conserved for PDE (1):

$$M_m(t) = \frac{1}{m}\int_0^1 [u(x,t)]^m \, dx, \quad \forall m \in \mathbb{N}\backslash\{0\}.$$

We define the Courant number as $\sigma = \frac{\Delta t}{\Delta x}$. The following finite-difference schemes are used to numerically approximate solutions of PDE (1):

FTFS: $u_j^{n+1} = u_j^n + \sigma(u_{j+1}^n - u_j^n)$,

FTCS: $u_j^{n+1} = u_j^n + \frac{\sigma}{2}(u_{j+1}^n - u_{j-1}^n)$,

FTBS: $u_j^{n+1} = u_j^n + \sigma(u_j^n - u_{j-1}^n)$,

BTCS: $u_j^{n+1} = u_j^n + \frac{\sigma}{2}(u_{j+1}^{n+1} - u_{j-1}^{n+1})$,

Lax-Friedrichs: $u_j^{n+1} = \frac{u_{j+1}^n + u_{j-1}^n}{2} + \frac{\sigma}{2}(u_{j+1}^n - u_{j-1}^n)$,

Lax-Wendroff: $u_j^{n+1} = u_j^n + \frac{\sigma}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{\sigma^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$.

For each of these six methods, answer the following questions:
Hint: For the BTCS method, first rewrite the finite-difference equation into a suitable form.
Hint: Do not be alarmed if some methods turn out to be unconditionally unstable.

(b) Take $\Delta x = \frac{1}{100}$ and $T = 2$ as fixed. Perform several numerical simulations with different values of the Courant number σ to illustrate the properties of each method. Comment on aspects such as accuracy, stability, numerical diffusion, numerical dispersion and/or oscillations. If applicable, determine values for Δt that result in stable and accurate numerical solutions. Compare your numerical results with the analytical solution, given by:

$$u(x,t) = u_0\big((x+t) \bmod 1\big).$$

(c) Plot relevant numerical solutions to support your answer to part (b). Compare with the exact solution at $t = 0$ and $t = T = 2$.

(d) Calculate the exact values of the conserved quantities:

$$m = 1: \quad \mathcal{M}(t) = \int_0^1 u(x,t)\, dx, \qquad \text{momentum}$$

and

$$m = 2: \quad \mathcal{E}(t) = \frac{1}{2}\int_0^1 [u(x,t)]^2\, dx, \qquad \text{energy.}$$

(e) Verify whether the numerical solutions of the six methods preserve the conserved quantities $\mathcal{M}(t)$ and $\mathcal{E}(t)$. *Hint: The degree of conservation may depend on the choice of time step* Δt

PART B (50 points)

Consider the following *infinite-order* PDE in one spatial dimension:

$$\frac{\partial u}{\partial t} = \alpha \sum_{k=0}^{\infty} \frac{\partial^k u}{\partial x^k}, \quad u(x,\cdot) \in \mathcal{C}^\infty(\mathbb{R}),\ t > 0,\ 0 < x < L,\ \alpha \in \mathbb{R}. \tag{2}$$

(a) Show that the solution $u(x,t)$ of PDE (2) must satisfy the *finite-order* PDE:

$$u_t = \alpha u + u_{xt}. \tag{3}$$

(b) We impose the boundary condition:

$$u_t(0,t) = u_t(L,t).$$

Apply the first step of the Method of Lines to PDE (3): use a central difference approximation for the spatial derivative and approximate the model as a system of ODEs in the form:

$$\dot{\vec{u}} = \mathcal{M}\vec{u}.$$

(c) Let $x \in [0,1]$, $t \in [0,20]$, and the initial condition be $u(x,0) = \sin(2\pi x)$. Using the second step of the Method of Lines, solve the system using the **Implicit Euler method** for $\alpha = -1$. Plot the numerical solutions for various step sizes Δx and Δt. Assess the numerical accuracy by comparing the approximations with the exact solution:

$$u(x,t) = e^{-\beta t}\left[-\sin(2\beta\pi t)\cos(2\pi x) + \cos(2\beta\pi t)\sin(2\pi x)\right],$$

where $\beta = \frac{1}{1+4\pi^2}$.

(d) Plot the eigenvalues of the matrix $\mathcal{M}$ and the stability region of the Implicit Euler method for $\Delta x = 0.05$. Do all eigenvalues fall within the stability region?

(e) Repeat parts (c) and (d) using the **Explicit Euler method**.

(f) Explain why both the Explicit and Implicit Euler methods are inappropriate for $\alpha = 1$.

(g) Derive an exact solution for the case $\alpha = 1$ using the Fourier Transform Method. Compare this solution with the analytical solution for $\alpha = -1$ and describe the key difference. You may use without proof that:

$$\lim_{x \to \pm\infty} u_t(x,t) = 0.$$