Its characteristic equation is

$$r^2 - \left[\frac{2}{1+\frac{h^2}{2}}\right] r + 1 = 0, \tag{2.3.29}$$

with roots

$$r_{\pm}(h) = \left[\frac{1}{1+\frac{h^2}{2}}\right]\left[1 \pm ih\sqrt{1+\frac{h^2}{4}}\right]. \tag{2.3.30}$$

Note that

$$r_+(h) = [r_-(h)]^*, \qquad h > 0, \tag{2.3.31}$$

$$|r_+(h)| = |r_-(h)| = 1; \tag{2.3.32}$$

consequently,

$$r_+ = r_-^* = e^{i\phi(h)}, \tag{2.3.33}$$

$$\tan\phi(h) = h\sqrt{1+\frac{h^2}{4}}. \tag{2.3.34}$$

Since

$$y_k = E(r_+)^k + E^*(r_+^*)^k, \tag{2.3.35}$$

where E is an arbitrary complex-valued constant, we conclude that all solutions to this discrete model oscillate with constant amplitude for $h > 0$.

The second example has a completely symmetric discrete expression for the linear term; it is

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + \frac{y_{k+1} + y_k + y_{k-1}}{3} = 0. \tag{2.3.36}$$

The corresponding characteristic equation is

$$r^2 - 2\left[\frac{1-\frac{h^2}{6}}{1+\frac{h^2}{3}}\right] r + 1 = 0, \tag{2.3.37}$$

with the roots

$$r_{\pm}(h) = \left[\frac{1}{1+\frac{h^2}{3}}\right]\left\{\left(1-\frac{h^2}{6}\right) \pm ih\sqrt{1+\frac{h^2}{12}}\right\}. \tag{2.3.38}$$

These roots have the following properties:

$$r_+(h) = [r_-(h)]^*, \qquad h > 0, \tag{2.3.39}$$

$$|r_+(h)| = |r_-(h)| = 1, \qquad h > 0, \tag{2.3.40}$$

$$r_+(h) = [r_-(h)]^* = e^{i\phi(h)},$$

$$\tan\phi(h) = \frac{h\sqrt{1+\frac{h^2}{12}}}{\left(1-\frac{h^2}{6}\right)}. \tag{2.3.41}$$

We conclude that, for $h > 0$, all solutions of Eq. (2.3.36) are oscillatory with constant amplitude.

In summary, we have seen that only the use of a discrete representation for the linear y term that is centered about the grid point t_k will give a discrete model that has oscillations with constant amplitude. Non-centered schemes allow the amplitude of the oscillations to either increase or decrease. The straightforward central difference scheme has the correct oscillatory behavior if $0 < h < 2$, while the two "symmetric" forms for y give oscillatory behavior with constant amplitude for all $h > 0$.

2.4 Logistic Differential Equation

The Logistic differential equation is

$$\frac{dy}{dt} = y(1-y). \tag{2.4.1}$$

Its exact solution can be obtained by the method of separation of variables which gives

$$y(t) = \frac{y_0}{y_0 + (1-y_0)e^{-t}}, \tag{2.4.2}$$

where the initial condition is

$$y_0 = y(0). \tag{2.4.3}$$

Figure 2.4.1 illustrates the general nature of the various solution behaviors. If $y_0 > 0$, then all solutions monotonically approach the stable fixed-point at $y(t) = 1$. If $y_0 < 0$, then the solution at first decreases to $-\infty$ at the singular point

$$t = t^* = \text{Ln}\left[\frac{1 + |y_0|}{|y_0|}\right], \tag{2.4.4}$$

after which, for $t > t^*$, it decreases monotonically to the fixed-point at $y(t) = 1$. Note that $y(t) = 0$ is an unstable fixed-point.

Our first discrete model is constructed by using a central difference scheme for the derivative:

$$\frac{y_{k+1} - y_{k-1}}{2h} = y_k(1 - y_k). \tag{2.4.5}$$

Since Eq. (2.4.5) is a second-order difference equation, while Eq. (2.4.1) is a first-order differential equation, the value of $y_1 = y(h)$ must be determined by some procedure. We do this by use of the Euler result [7, 8, 9]

$$y_1 = y_0 + hy_0(1 - y_0). \tag{2.4.6}$$

A typical plot of the numerical solution to Eq. (2.4.5) is shown in Figure 2.4.2. This type of plot is obtained for any value of the step-size. An understanding of this result follows from a linear stability analysis of the two fixed points of Eq. (2.4.5).

First of all, note that Eq. (2.4.5) has two constant solutions or fixed-points. They are

$$y_k = \bar{y}^{(0)} = 0, \qquad y_k = \bar{y}^{(1)} = 1. \tag{2.4.7}$$

To investigate the stability of $y_k = \bar{y}^{(0)}$, we set

$$y_k = \bar{y}^{(0)} + \epsilon_k, \qquad |\epsilon_k| \ll 1, \tag{2.4.8}$$

substitute this result into Eq. (2.4.5) and neglect all but the linear terms. Doing this gives

$$\frac{\epsilon_{k+1} - \epsilon_{k-1}}{2h} = \epsilon_k. \tag{2.4.9}$$

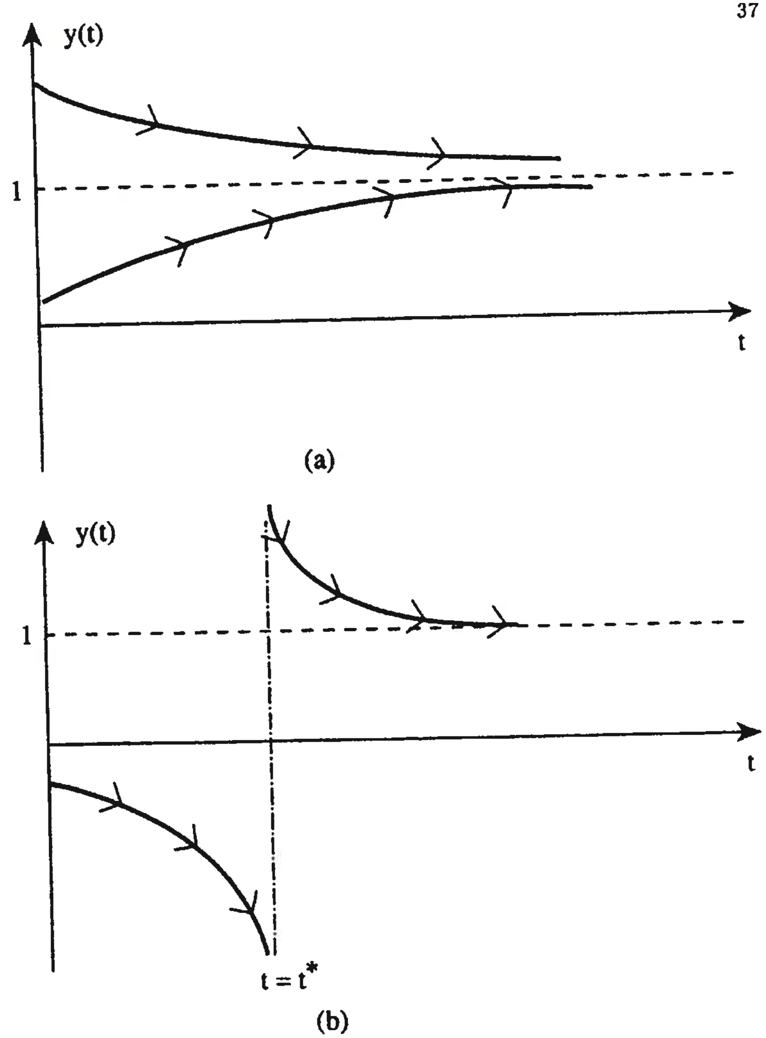


Figure 2.4.1. Solutions of the logistic differential equation. (a) $y_0 > 0$. (b) $y_0 < 0$.

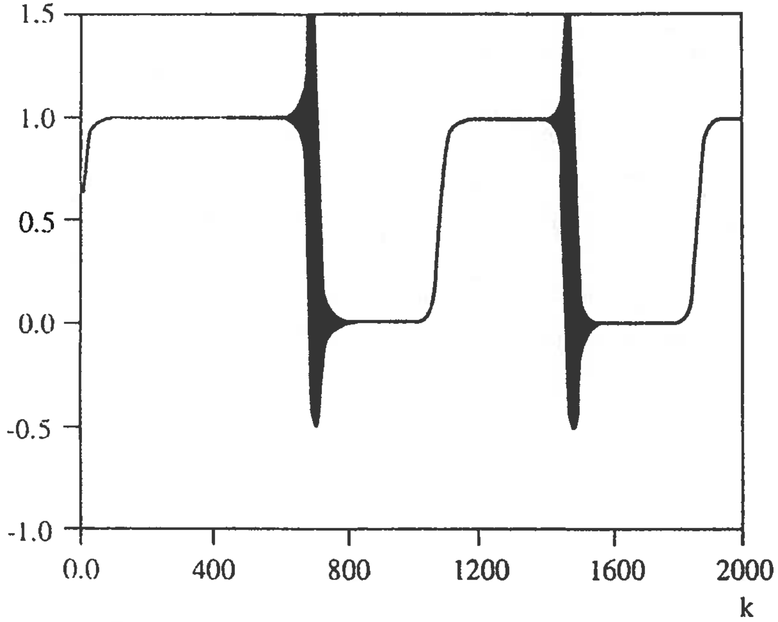


Figure 2.4.2. Typical plot for a central difference scheme model of the logistic differential equation: $y_0 = 0.5$, $h = 0.1$.

$$\frac{y_{k+1} - y_{k-1}}{2h} = y_k(1 - y_k).$$

The solution to this second-order difference equation is

$$\epsilon_k = A(r_+)^k + B(r_-)^k, \tag{2.4.10}$$

where A and B are arbitrary, but, small constants; and

$$r_\pm(h) = h \pm \sqrt{1 + h^2}. \tag{2.4.11}$$

From Eq. (2.4.11), it can be concluded that the first term on the right-side of Eq. (2.4.10) is exponentially increasing, while the second term oscillates with an exponentially decreasing amplitude.

A small perturbation to the fixed-point at $\bar{y}^{(1)} = 1$ can be represented as

$$y_k = \bar{y}^{(1)} + \eta_k, \qquad |\eta_k| \ll 1. \tag{2.4.12}$$

The linear perturbation equation for η_k is

$$\frac{\eta_{k+1} - \eta_{k-1}}{2h} = -\eta_k, \tag{2.4.13}$$

whose solution is

$$\eta_k = C(S_+)^k + D(S_-)^k, \tag{2.4.14}$$

where C and D are small arbitrary constants, and

$$S_\pm(h) = -h \pm \sqrt{1 + h^2}. \tag{2.4.15}$$

Thus, the first term on the right-side of Eq. (2.4.14) exponentially decreases, while the second term oscillates with an exponentially increasing amplitude.

Putting these results together, it follows that the central difference scheme has exactly the same two fixed-points as the Logistic differential equation. However, while $y(t) = 0$ is (linearly) stable and $y(t) = 1$ is (linearly) unstable for the differential equation, both fixed-points are linearly unstable for the central difference scheme. The results of the linear stability analysis, as given in Eqs. (2.4.10) and

(2.4.14), explain what is shown by Figure 2.4.2. For initial value y_0, such that $0 < y_0 < 1$, the values of y_k increase and exponentially approach the fixed-point $\bar{y}^{(1)} = 1$; y_k then begins to oscillate with an exponentially increasing amplitude about $\bar{y}^{(1)} = 1$ until it reaches the neighborhood of the fixed-point $\bar{y}^{(0)} = 0$. After an initial exponential decrease to $\bar{y}^{(0)} = 0$, the y_k value then begin their increase back to the fixed-point at $\bar{y}^{(1)} = 0$.

It has been shown by Yamaguti and Ushiki [10] and by Ushiki [11] that the central difference scheme allows for the existence of chaotic orbits for all positive time-steps for the Logistic differential equation. Additional work on this problem has been done by other researchers including Sanz-Serna [12] and Mickens [13]. The major conclusion is that the use of the central difference scheme

$$\frac{y_{k+1} - y_{k-1}}{2h} = f(y_k), \tag{2.4.16}$$

for the scalar first-order differential equation

$$\frac{dy}{dt} = f(y) \tag{2.4.17}$$

forces all the fixed-points to become unstable [13]. Consequently, the central difference discrete derivative should never be used for this class of ordinary differential equation.

However, before leaving the use of the central difference scheme, let us consider the following discrete model for the Logistic equation:

$$\frac{y_{k+1} - y_{k-1}}{2h} = y_{k-1}(1 - y_{k+1}). \tag{2.4.18}$$

Our major reason for studying this model is that an exact analytic solution exists for Eq. (2.4.18). Observe that the function

$$f(y) = y(1 - y) \tag{2.4.19}$$

is modeled locally on the lattice in Eq. (2.4.5), while it is modeled nonlocally in Eq. (2.4.18), i.e., at lattice points $k-1$ and $k+1$.

The substitution

$$y_k = \frac{1}{x_k}, \tag{2.4.20}$$

transforms Eq. (2.4.18) to the expression

$$x_{k+1} - \left(\frac{1}{1+2h}\right)x_{k-1} = \frac{2h}{1+2h}. \tag{2.4.21}$$

Note that Eq. (2.4.18) is a nonlinear, second-order difference equation, while Eq. (2.4.21) is a linear, inhomogeneous equation with constant coefficients. Solving Eq. (2.4.21) gives the general solution

$$x_k = 1 + [A + B(-1)^k](1+2h)^{-k/2}, \tag{2.4.22}$$

where A and B are arbitrary constants. Therefore, y_k is

$$y_k = \frac{1}{1 + [A + B(-1)^k](1+2h)^{-k/2}}. \tag{2.4.23}$$

For y_0 such that $0 < y_0 < 1$, and y_1 selected such that $y_1 = y_0 + hy_0(1 - y_0)$, the solutions to Eq. (2.4.23) have the structure indicated in Figure 2.4.3. Observe that the numerical solution has the general properties of the solution to the Logistic differential equation, see Figure 2.4.1, except that small oscillations occur about the smooth solution.

The direct forward Euler discrete model for the Logistic differential equation is

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_k). \tag{2.4.24}$$

This first-order difference equation has two constant solutions or fixed-points at $\bar{y}^{(0)} = 0$ and $\bar{y}^{(1)} = 1$. Perturbations about these fixed-points, i.e.,

$$y_k = \bar{y}^{(0)} + \epsilon_k = \epsilon_k, \qquad |\epsilon_k| \ll 1, \tag{2.4.25}$$

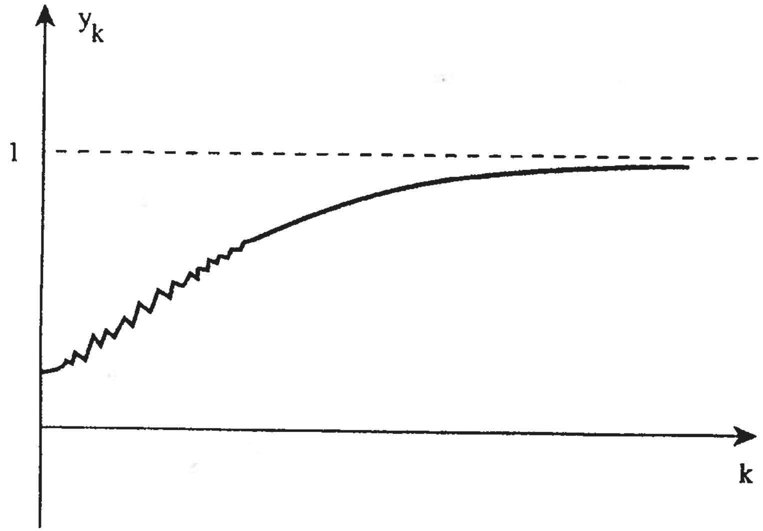


Figure 2.4.3. A trajectory for the central difference scheme
$$\frac{y_{k+1}-y_{k-1}}{2h}=y_{k-1}(1-y_{k+1}).$$

$$y_k = \bar{y}^{(1)} + \eta_k = 1 + \eta_k, \qquad |\eta_k| \ll 1, \tag{2.4.26}$$

give the following solutions for ϵ_k and η_k:

$$\epsilon_k = \epsilon_0(1+h)^k, \tag{2.4.27}$$

$$\eta_k = \eta_0(1-h)^k. \tag{2.4.28}$$

The expression for ϵ_k shows that $\bar{y}^{(0)}$ is unstable for all $h > 0$; thus, this discrete scheme has the same linear stability property as the differential equation for all $h > 0$. However, the linear stability properties of the fixed-point $\bar{y}^{(1)}$ depend on the value of the step-size. For example:

(i) $0 < h < 1 : \bar{y}^{(1)}$ is linearly stable; perturbations decrease exponentially.

(ii) $1 < h < 2 : \bar{y}^{(1)}$ is linearly stable; however, the perturbations decrease exponentially with an oscillating amplitude.

(iii) $h > 2 : \bar{y}^{(1)}$ is linearly unstable; the perturbations oscillate with an exponentially increasing amplitude.

Our conclusion is that the forward Euler scheme gives the correct linear stability properties only if $0 < h < 1$. For this interval of step-size values, the qualitative properties of the solutions to the differential and difference equations are the same. Consequently, for $0 < h < 1$, there are no numerical instabilities.

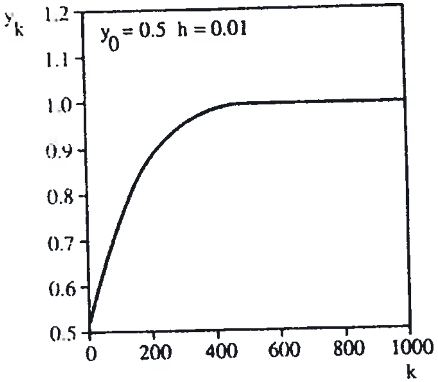
Figure 2.4.4 presents three numerical solutions for the forward Euler scheme given by Eq. (2.4.24). In all three cases the initial condition is $y_0 = 0.5$. The step-sizes are $h = 0.01$, 1.5 and 2.5.

Finally, it should be stated that the change of variables

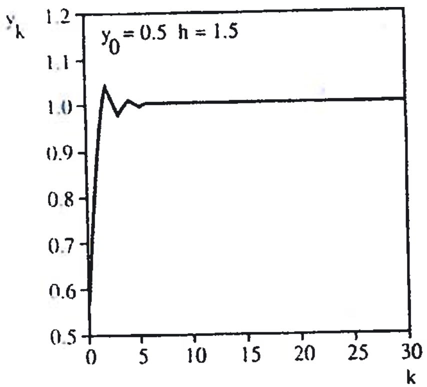
$$z_k = \left(\frac{h}{1+h}\right) y_k, \qquad \lambda = 1 + h, \tag{2.4.29}$$

when substituted into Eq. (2.4.24) gives the famous Logistic difference equation [7, 14, 15]

$$z_{k+1} = \lambda z_k(1 - z_k). \tag{2.4.30}$$



(a)

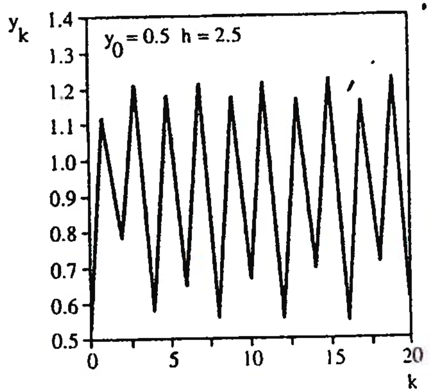


(b)

Figure 2.4.4. The forward Euler scheme

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_k).$$

(a) $y_0 = 0.5$, $h = 0.01$. (b) $y_0 = 0.5$, $h = 1.5$.



(c)

Figure 2.4.4. The forward Euler scheme

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_k).$$

(c) $y_0 = 0.5$, $h = 2.5$.

Depending on the value of the parameter λ, this equation has a host of solutions with various periods, as well as chaotic solutions [16, 17].

Our next model of the Logistic differential equation is constructed by using a forward Euler for the first-derivative and a nonlocal expression for the function $f(y) = y(1-y)$. This model is

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_{k+1}). \tag{2.4.31}$$

This first-order, nonlinear difference equation can be solved exactly by using the variable change

$$y_k = \frac{1}{x_k}, \tag{2.4.32}$$

to obtain

$$x_{k+1} - \left(\frac{1}{1+h}\right)x_k = \frac{h}{1+h}, \tag{2.4.33}$$

whose general solution is

$$x_k = 1 + A(1+h)^{-k}, \tag{2.4.34}$$

where A is an arbitrary constant. Imposing the initial condition

$$x_0 = \frac{1}{y_0}, \tag{2.4.35}$$

gives

$$A = \frac{1 - y_0}{y_0}, \tag{2.4.36}$$

and

$$y_k = \frac{y_0}{y_0 + (1 - y_0)(1+h)^{-k}}. \tag{2.4.37}$$

Examination of Eq. (2.4.37) shows that, for $h > 0$, its qualitative properties are the same as the corresponding exact solution to the Logistic differential equation, namely, Eq. (2.4.2). Hence, the forward Euler, nonlocal discrete model has no numerical instabilities for any step-size. Figure 2.4.5 gives numerical solutions using

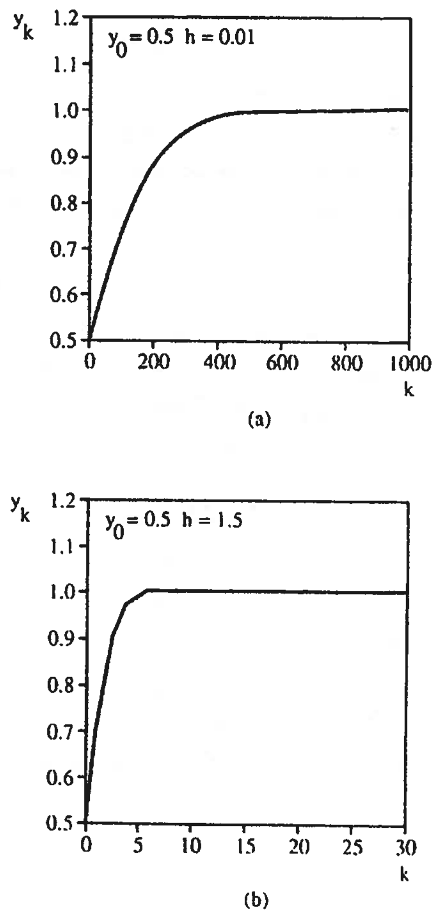
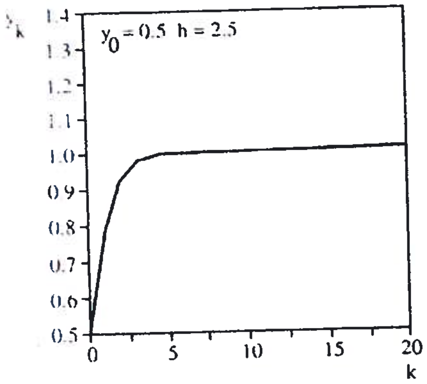


Figure 2.4.5. Numerical solutions of

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_{k+1}).$$

(a) $y_0 = 0.5$, $h = 0.01$. (b) $y_0 = 0.5$, $h = 1.5$.



(c)

Figure 2.4.5. Numerical solutions of

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_{k+1}).$$

(c) $y_0 = 0.5$, $h = 2.5$.

Eq. (2.4.31) for three step-sizes. Note that Eq. (2.4.31) can be written in explicit form

$$y_{k+1} = \frac{(1+h)y_k}{1 + hy_k}. \tag{2.4.38}$$

Our last discrete model for the Logistic differential equation is based on a second-order Runge-Kutta method [8, 9]. This technique gives for the first order scalar equation

$$\frac{dy}{dt} = f(y), \tag{2.4.39}$$

the discrete result

$$\frac{y_{k+1} - y_k}{h} = \frac{f(y_k) + f[y_k + hf(y_k)]}{2}. \tag{2.4.40}$$

Applying this to the Logistic equation, where $f(y) = y(1-y)$, gives

$$y_{k+1} = \left[1 + \frac{(2+h)h}{2}\right] y_k - \left[\frac{(2+3h+h^2)h}{2}\right] y_k^2 + (1+h)h^2 y_k^3 - \left(\frac{h^3}{2}\right) y_k^4. \tag{2.4.41}$$

This first-order, nonlinear difference equation has four fixed-points. They are located at

$$\bar{y}^{(0)} = 0, \qquad \bar{y}^{(1)} = 1, \tag{2.4.42}$$

$$\bar{y}^{(2,3)} = \left(\frac{1}{2h}\right)\left[(2+h) \pm \sqrt{h^2 - 4}\right]. \tag{2.4.43}$$

The first two fixed-points, $\bar{y}^{(0)}$ and $\bar{y}^{(1)}$, correspond to the two fixed-points of the Logistic differential equation. The other two fixed-points, $\bar{y}^{(2)}$ and $\bar{y}^{(3)}$, are spurious fixed-points and are introduced by the second-order Runge-Kutta method. Note that for $h \leq 2$, the fixed-points $\bar{y}^{(2)}$ and $\bar{y}^{(3)}$ are complex conjugates of each other; while for $h \geq 2$, all fixed-points are real. Figure 2.4.6 gives a plot of all the fixed-points as a function of the step-size h.

For $0 < h < 2$, there are only two real fixed-points, namely, $\bar{y}^{(0)} = 0$ and $\bar{y}^{(1)} = 1$. The first is linearly unstable and the second is linearly stable. All numerical solutions of Eq. (2.4.41), with $y_0 > 0$, thus approach $\bar{y}^{(1)}$ as $k \to \infty$. However, for $h > 2$, there exists four real fixed-points. Their order and linear

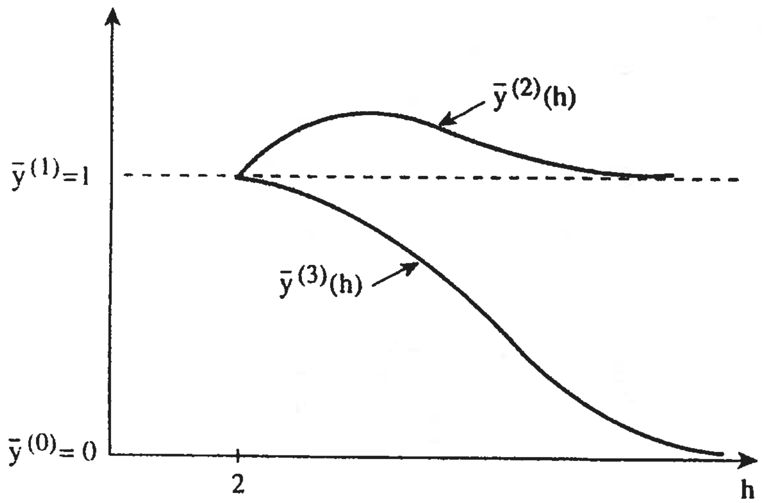


Figure 2.4.6. Plot of the fixed-points of the 2nd-order Runge-Kutta method for the logistic differential equation. Only the spurious fixed-points depend on h.

stability properties are indicated below where U and S, respectively, mean linearly unstable and linearly stable:

$$\bar{y}^{(0)} < \bar{y}^{(3)}(h) < \bar{y}^{(1)} < \bar{y}^{(2)}(h)$$
$$U \quad S \quad U \quad S.$$

These results and Eq. (2.4.43) predict that at a step-size of $h = 2.5$, if the initial value y_0 is selected so that $0 < y_0 < 1$, then the numerical solution of Eq. (2.4.41) will converge to the value 0.6. The validity of this prediction is shown in Figure 2.4.7(c). This figure also gives numerical solutions for several other step-sizes.

The application of the second-order Runge-Kutta method illustrates the generation of numerical instabilities that arise from the creation of additional spurious fixed-points.

Comparing the five finite-difference schemes that were used to model the Logistic differential equation, the nonlocal forward Euler method clearly gave the best results. For all values of the step-size it has solutions that are in qualitative agreement with the corresponding solutions of the differential equation. The other discrete models had, for certain values of step-size, numerical instabilities.

2.5 Unidirectional Wave Equation

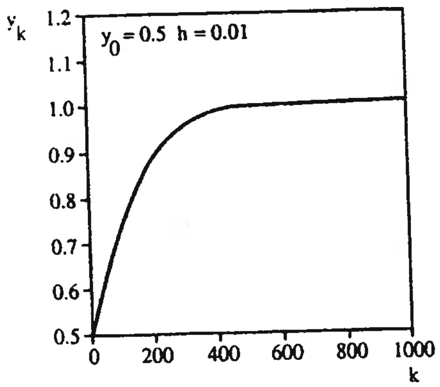
The one-way or unidirectional wave equation is [10]

$$u_t + u_x = 0, \qquad u(x,0) = f(x), \tag{2.5.1}$$

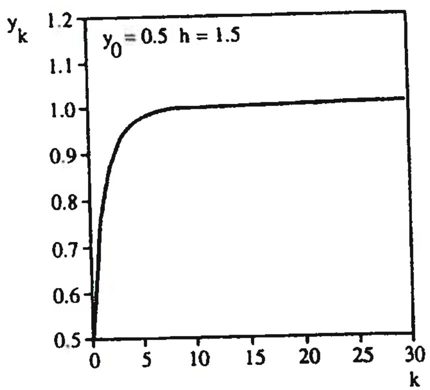
where the initial profile function $f(x)$ is assumed to have a first derivative. The solution to the initial value problem of Eq. (2.5.1) is

$$u(x,t) = f(x-t). \tag{2.5.2}$$

This represents a waveform moving to the right with unit velocity.

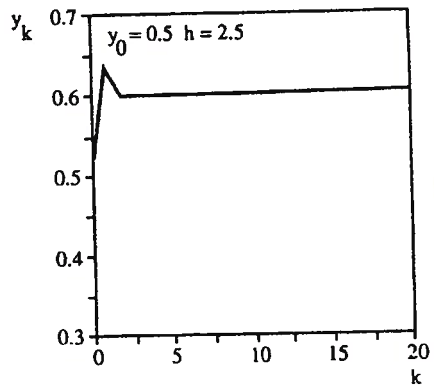


(a)

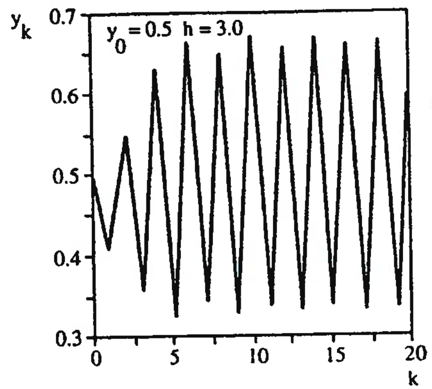


(b)

Figure 2.4.7. Numerical integration of the logistic equation by a 2nd-order Runge-Kutta method.
(a) $y_0 = 0.5$, h = 0.01. (b) $y_0 = 0.5$, h = 1.5.



(c)



(d)

Figure 2.4.7. Numerical integration of the logistic equation by a 2nd-order Runge-Kutta method.
(c) $y_0 = 0.5$, h = 2.5. (d) $y_0 = 0.5$, h = 3.0.