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An efficient nonlinear multigrid scheme for 2D boundary value problems

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ABSTRACT

In this article, a two-dimensional nonlinear boundary value problem which is strongly related to the well-known Gelfand–Bratu model is solved numerically. The numerical results are obtained by employing three different numerical strategies namely: finite difference based method, a Newton multigrid method and a nonlinear multigrid full approximation storage (FAS). We are able to handle the difficulty of unstable convergence behaviour by using MINRES method as a relaxation smoother in multigrid approach with an appropriate sinusoidal approximation as an initial guess. A comparison, in terms of convergence, accuracy and efficiency among the three numerical methods demonstrate an improvement for the values of $\lambda \in (0, \lambda_c]$. Numerical results illustrate the performance of the proposed numerical methods wherein FAS-MG method is shown to be the most efficient. Further, we present the numerical bifurcation behaviour for two-dimensional Gelfand-Bratu models and find new multiplicity of solutions in the case of a quadratic and cubic approximation of the nonlinear exponential term. Numerical experiments confirm the convergence of the solutions for different values of λ and prove the effectiveness of the nonlinear FAS-MG scheme.

1. Introduction

In this article, we consider the nonlinear boundary value problem (BVP):

$$\begin{aligned} &\Delta u + \lambda\, f(u) = 0, \quad (x, y) \in \Omega, \\ &u|_{\partial\Omega} = 0, \end{aligned} \tag{1.1}$$

where $u = u(x, y)$, $\lambda > 0$ and $\Omega = [0, 1]^2$ with boundary $\partial\Omega$. For the special choice $f(u) = e^u$ the BVP (1.1) is a classical two-dimensional Gelfand–Bratu (GB) equation [1–3]. The GB model appears in a number of applications such as thermo-electro-hydrodynamics models [4], nanotechnology, radioactive heat transfer, chemical reaction theory, fuel ignition model [5], thermal reaction processes, the Chandrasekhar model [6], and in elasticity theory [7].

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In the past two decades, several numerical methods have been proposed for the Gelfand–Bratu problem including multilevel continuation techniques [8], multilevel adaptive methods [9], variational iterative methods [10], explicit radial solutions [11], iterative quadrature methods [12], the WR-method [13], nonstandard compact schemes [14], wavelet methods [15], arc-length continuation with multigrid [16], a Sinc–Galerkin method [17], pseudospectral methods [18], techniques for the calculation of turning points [19], the variational Iteration method [20]. Most numerical solutions of these methods converge only to the first (lower) solution for $0 < \lambda < \lambda_c$. Moreover, the mentioned methods lose significantly their accuracy when λ approaches the critical value λ_c. In [21], a finite difference method with a nonlinear solver is employed for a 2D Bratu model, in which the author only showed numerical solutions for a grid size $h = 1/10$.

It is well known that the multigrid (MG) idea is based on two principles: an error smoothing on fine grids and followed by coarse grid corrections. MG schemes are efficient and fast elliptic equations solvers, in general. One approach is to construct an MG scheme on the basis of a global Newton linearization and then solve the resulting system with a linear multigrid scheme [22] or with a Newton- Krylov method. Global linearization techniques that use multigrid as a linear solver or as a preconditioner for Krylov methods are presented in [23–25]. Alternatively, MG can be applied directly to nonlinear elliptic equations without the use of a global linearization. This is known as the full approximation storage (FAS) [9]. In FAS-MG, linearization of the nonlinear problem is treated locally in the relaxation step on different grid sizes.

In the literature, only a few multigrid schemes could obtain both solutions of the Bratu problem. The convergence of the first (lower) solution can be successfully obtained by the classical FAS-MG scheme with a Gauss-Seidel smoother. However, the convergence rate of this multigrid solver is very slow and the method could even diverge. This unstable convergence behaviour could be caused by the diagonal element in the exponential form and the linearized system may then become indefinite as the diagonal dominance of the Jacobian matrix is lost [26]. The Jacobian matrix depends on the solution values, the parameter λ and the grid size h. In that case, Gauss-Seidel is no longer the right choice as a smoother. Hackbusch [27] applied two versions of the nonlinear multigrid scheme, FAS-MG and a nonlinear multigrid method (NMGM). He successfully found the upper solution, but only for large grid sizes: $h = 1/4$ and $h = 1/8$. Washio and Oosterlee [28] presented a nonlinear Krylov acceleration method as the outer iteration with a MG preconditioner for the 2D Gelfand–Bratu problem for small values of λ with different choices of the smoother. They mentioned that there is no numerical second solution exists on the grid of size $h = 1/4$ for small values of λ. So the size $h = 1/8$ was chosen as the coarsest grid fine enough for a second solution to exist.

To handle the unstable convergence problem, in a previous numerical MG study [29], two Krylov methods as smoothers were employed for solving the Bratu problem. For small values of the bifurcation parameter λ, the Newton-MG method, FMG-FAS and FMG-NMGM with the two Krylov methods as smoothers and u_c as an initial guess, were used to obtain both solutions of the Bratu problem in [29]. This MG was still unable to achieve the desired accuracy for small grid sizes h.

The present work is an extension of our 1D case [30]. We are aiming to extend these numerical results to the 2D Gelfand–Bratu problem. Particularly, one of the goals is to obtain new types of solutions, as we did in the 1D case [30]. To achieve this, we propose three numerical methods: a finite difference (FD) based method, Newton-MG and FAS-MG, to compute all possible solutions of the given nonlinear BVP (1.1). These methods successfully handle the mentioned convergence difficulties. In particular, we are able to obtain the upper solution of the two-dimensional Bratu problem for small grid sizes h and for small values of λ. In the FD-method, we use the Matlab nonlinear solver *fsolve* to solve the nonlinear system of equations and in MG methods, MINRES is used as a relaxation smoother. Further, we employ the sinusoidal function as an initial approximation for the three proposed numerical methods. Numerical implementation of these methods is provided in more detail, see Section 3. The main contribution of the present work is computation of more accurate solutions and confirmation of the convergence of the second branch of solutions (the upper solutions) for small values of λ. We obtain the desired accuracy on decreasing value of grids size h with the help of proposed numerical strategies. The problem of the indefinite Jacobian matrix is solved successfully for large values of the solution, which correspond with the upper solution in the model. To achieve this, we use the MINRES method as a smoothing relaxation in MG methods and a sinusoidal function as an initial guess. Moreover, we present new types of solutions: periodic and semi-periodic solutions. For special cases of the Gelfand–Bratu problem we also obtain multiple solutions by approximating the nonlinear term. We conjecture that the number of solutions may vary between 0 and ∞ depending on the model parameter λ. The convergence of a numerical method to find new multiplicities of solutions, especially for small values of λ on a large number of grid points, is of great importance. We present the numerical convergence of these new multiple solutions by employing the proposed numerical methods.

The organization of this article is follows. In Section 2, we present three numerical methods: a finite difference(FD) based method, Newton-MG and FAS-MG, respectively for computing the numerical solutions of the two-dimensional GB model. Numerical experiments in Section 3, illustrate the bifurcation behaviour of the GB model and demonstrate the convergence of all types of solutions: lower, upper periodic and semi-periodic for different values of λ. Further, numerical results show the performance of the proposed methods. Finally, concluding remarks are presented in Section 4.

2. A description of the three numerical methods

To compute numerical solutions of the nonlinear boundary value problem (1.1), we present three numerical methods: a finite difference based method, Newton-MG and FAS-MG. More details of these numerical methods will be discussed in the following sections.

2.1. Method 1: a central finite difference approximation with fsolve.m

Our first numerical method computes solutions with a central finite difference (FD) scheme. On a uniform spatial grid on the domain $[0, 1]^2$ we generate the grid points by:

$$x_i = i\,h, \quad y_j = j\,h, \quad i, j = 0, 1, 2, \ldots J, \quad \text{with} \quad h = \frac{1}{J}.$$

We denote by $u_{i,j}$ the discrete approximation of the exact (unknown) values $u(x_i, y_j)$ Applying a second-order central finite difference approximation on the uniform grid Ω_h, yields:

$$\underbrace{\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}}_{\mathcal{A}\,\mathrm{u}} + \lambda\, f(u_{i,j}) = 0, \quad i, j = 1, 2, \ldots, J-1, \tag{2.1}$$

with the Dirichlet boundary conditions: $u_{0,j} = u_{J,j} = u_{i,0} = u_{i,J} = 0$. The FD scheme at the grid points (x_i, y_j) makes use of the five grid points, (x_{i-1}, y_j), (x_{i+1}, y_j), (x_i, y_{j-1}), (x_i, y_{j+1}) and (x_i, y_j) and the corresponding coefficient matrix $\mathcal{A}$ becomes block-tridiagonal:

$$\mathcal{A} = \frac{1}{h^2}\begin{bmatrix} \mathcal{B} & \mathcal{I} & O & O \\ \mathcal{I} & \mathcal{B} & \mathcal{I} & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \mathcal{I} & \mathcal{B} \end{bmatrix}_{(J-1)^2 \times (J-1)^2} \quad \textit{with} \quad \mathcal{B} = \begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & -4 \end{bmatrix}_{(J-1)\times(J-1)},$$

where $\mathcal{I}$ and O are the $(J-1) \times (J-1)$ identity and zero matrices, respectively. In the next step, we solve the discretized system of algebraic equations (2.1) to obtain the approximate solution at each grid point, Finally, the resulting nonlinear system is solved iteratively by applying the nonlinear solver from matlab: *fsolve.m*. This nonlinear solver starts at initial guess u^0 and finds the approximate solution of nonlinear system of equations at certain accuracy. We set the termination tolerance to be: $tol = 10^{-10}$ and the initial approximation is chosen as (see also in the 1D case [30]).

$$u_{initial} = u^0 = \alpha \sin(k\pi x_i) \sin(k\pi y_j), \tag{2.2}$$

with an amplitude α and frequency k. These numerical parameters need to be specified for each experiment separately. Knowing that the maximum solution u_c, the lower solution requires $\alpha < u_c(\frac{1}{2}, \frac{1}{2})$ and the upper solution requires $\alpha > u_c(\frac{1}{2}, \frac{1}{2})$ for numerical experiments. The other parameters value specifies $k = 1$ for lower solutions and $k \geq 1$ for upper solutions. This approach is simple and easy to implement numerically for solving the nonlinear system of Eqs. (2.1). However the method is not efficient particularly for small values of λ and increasing the problem size. In such situation, convergence of the solution becomes very slow and sometimes prevents obtaining the set accuracy. To handle this difficulty, we choose the numerical parameters α and k very carefully, as *fsolve* is very sensitive to the initial guess, especially for small values of λ on an increasing number of grid points.

Multigrid (MG) schemes are efficient and fast elliptic equation solvers, in general. To enhance the accuracy and efficiency, we propose to choose the same initial guess for MG methods as used in the current approach (FD method).

2.2. Multigrid methods

Multigrid methods have been developed for solving nonlinear partial differential equations (PDE) efficiently [31–34]. For solving the discretized system of the nonlinear boundary value problem (1.1), we discuss two multigrid approaches. The first approach is a *Newton-multigrid* method, in which global linearization is naturally applied, like by Newton's method. After having linearized the PDE problem, in each linearization step, the linear multigrid method can be used for solving the Jacobian system. The second approach is known as the *Full Approximation Storage* (FAS), in which the multigrid method is applied directly to the given nonlinear problem. In the FAS approach, instead of solving an equation for corrections of the solution on a coarse grid, the original equation is prescribed and full solutions are also computed on coarse grids. In the next two subsections we will describe both multigrid methods in more detail.

2.2.1. Method 2: A newton-multigrid method (newton-MG)

Consider the nonlinear system of Eq. (2.1) on fine grid h to be solved as:

$$N_h(\mathrm{u}_h) = 0 \tag{2.3}$$

where $N(.)$ denotes the discrete version of the nonlinear operator, and is defined as

$$N(\mathrm{u}) = \mathcal{A}\mathrm{u} + \lambda f(\mathrm{u}).$$

The residual on fine grid h can be rewritten as:

$$r_h(\mathrm{u}_h) = 0 - N_h(\mathrm{u}_h)$$

The Newton iteration k, for solving equation (2.3), is characterized by

$$\mathrm{u}_h^{k+1} = \mathrm{u}_h^k + \underbrace{\left(\mathbb{J}(\mathrm{u}_h^k)\right)^{-1} r_h(\mathrm{u}_h^k)}_{e_h^k}, \qquad k = 0, 1, 2, 3, \ldots \tag{2.4}$$

The current approximation of the nonlinear system of equations (2.3) can be obtained by solving (2.4). For this, the following linear system is to be solved for e_h^k as:

$$\mathbb{J}\left(\mathrm{u}_h^k\right) e_h^k = r_h\left(\mathrm{u}_h^k\right), \tag{2.5}$$

where $\mathbb{J}(\mathrm{u}_h^k) = N_h^{'}(\mathrm{u}_h^k)$, is the Jacobian matrix of the nonlinear system and $r_h(\mathrm{u}_h^k) = -N_h(\mathrm{u}_h^k)$. Within each Newton iteration, we need to calculate the exact Jacobian matrix $\mathbb{J}(u_h)$ and its inverse. This process is, of course, time consuming, especially for large scale systems. This problem has led to the development of different variants of Newton's method. Solving (2.5) approximately by a chosen iterative method, is called an *Inexact Newton Method*. The most widely used inexact Newton method, is the Newton multigrid scheme, that makes use of an iterative method to solve the Jacobian system. The Jacobian matrix for (2.5) of the nonlinear system of Eq. (2.3) reads:

$$\mathbb{J}(\mathrm{u}_h) = \begin{bmatrix} \frac{4}{h^2} & -\frac{1}{h^2} & 0 & 0 \\ -\frac{1}{h^2} & \frac{4}{h^2} & -\frac{1}{h^2} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -\frac{1}{h^2} & \frac{4}{h^2} \end{bmatrix}_{(J-1)\times(J-1)} - \lambda * diag(f(u_{i,j})). \tag{2.6}$$

To reduce the time of calculating the Jacobian matrix, it is possible to divide the steps into two parts. In (2.6), the first part is independent of the solution u. Therefore, it needs to be calculated only once, whereas the second part uses a diagonal matrix that depends on the solution u and needs to be calculated in each iteration. At each step of the Newton-MG method, we employ Newton's method for the outer iteration and a linear multigrid method for the inner iteration for solving (2.5).

Newton-MG, the linear MG approach, initially solves Eq. (2.5) by an iterative method (a relaxation smoother). We use the MINRES method as a smoothing relaxation in Newton-MG. We apply a Newton-MG-MINRES method with *Nested* iterations for computing the numerical results of problem (1.1). It is well known that the Newton method converges if the initial approximation is close enough to the desired solution. For this, the initial approximation (2.2) is used for the Newton-MG method. The numerical parameters α and k need to be specified for each numerical experiment (see Section 3). Numerical results indicate the effectiveness of Newton-MG with MINRES and an appropriate initial guess.

2.2.2. Method 3: A nonlinear multigrid method

Although the Newton-multigrid method is an effective method, it does not make use of multigrid ideas directly to solve the discretized nonlinear system of equations. Much more memory is required for the Newton-MG method for the storage of the fine and coarse level Jacobian values. In the following approach, the multigrid scheme is applied directly to the full nonlinear problem. The nonlinear multigrid algorithm is called *full approximation storage* (FAS) (see also Brandt [9]). In the numerical scheme the nonlinear problem is solved at each level of the grid via multigrid cycles. In this work, we consider a discretized nonlinear system of Eq. (2.3) on the original fine grid h. Two grids, h and $2h$ are used to simplify notation for fine and coarse grids. In FAS-MG, suppose u_h is the approximate solution on the fine grid. The solution process may start with an iterative relaxation on the fine grid. Then the fine grid residual is transferred by a restriction operator R_h^{2h} to a coarse grid. Solve the coarse grid error problem to eliminate the corresponding error. This error is interpolated by linear interpolation operator I_{2h}^h to the fine grid and further use to correct the fine grid approximation. In the present work, we choose to linearize the continuous problem and then apply MINRES method as a smoother for the discretized linearized equation, see also [28,29]. The linearized equation for the nonlinear boundary-value problem (1.1) reads:

$$\Delta \mathrm{u} + \lambda\, f(\tilde{\mathrm{u}})\, \mathrm{u} = \lambda\, (\tilde{\mathrm{u}} - 1)\, f(\tilde{\mathrm{u}}),$$

where $\tilde{\mathrm{u}}$ is the current approximate solution of u for the problem (1.1). Assume that step ν_1 and step ν_2 are the pre-smoothing and post-smoothing steps, respectively. The complete FAS-MG two grid V-Cycle is given in Algorithm 1:

Algorithm 1: FAS V-Cycles.

1. Relax ν_1 times on $N_h(\mathrm{u}_h) = 0$, using nonlinear smoother with initial guess u_h^0, to compute u_h.
2. Compute the residual on the fine grid: $r_h = -N_h(\mathrm{u}_h)$.
3. Restrict the residual: $r_{2h} = R_h^{2h} r_h$.
4. Inject the solution $\mathrm{u}_{2h}^{init} = I_h^{2h}\mathrm{u}_h$.
5. If coarsest grid then solve $N_{2h}(\mathrm{u}_{2h}) = N_{2h}(\mathrm{u}_{2h}^{init}) - r_{2h}$ directly, else, call FAS scheme recursively.
6. Calculate the coarse grid error $e_{2h} = \mathrm{u}_{2h} - \mathrm{u}_{2h}^{init}$.
7. Interpolate the coarse grid error approximation to the fine grid: $e_h = I_{2h}^h e_{2h}$.
8. Correct the fine grid approximation: $\mathrm{u}_h = \mathrm{u}_h + e_h$.
9. Relax ν_2 times on $N_h(\mathrm{u}_h) = 0$, using nonlinear smoother with initial guess u_h (from step 8).

In FAS-MG, each grid problem will itself be a nonlinear differential equation at each grid level. In the present work, for inter-grid transfer, we use *bilinear* interpolation I_h^{2h} and *full-weighting* restriction R_h^{2h} operators. For the numerical experiments, we choose the initial guess (2.2) and a MINRES method as the smoothing relaxation as used in the Newton-MG method. Numerical results in Section 3 show the improvement in efficiency and the accuracy of the FAS-MG method.

3. Numerical experiments

In this section, we present numerical experiments for the three numerical methods applied to the BVP (1.1). For the computation of numerical solutions of GB model (1.1), we employ for each experiment: the FD scheme with nonlinear solver *fsolve*, Newton-MG within *nested* iterations and in FAS-MG, we use the V(2, 2) cycle. In both MG approaches, the MINRES method is used as a relaxation smoother (both for pre and post relaxation). We choose (2.2) as an initial approximation with the numerical parameters α and k for numerical computations of solutions. Knowing that the maximum solution $u_c \approx 1.392422635$, the lower solution requires $\alpha < u_c$ with the smaller values for small value of λ. The upper solution requires $\alpha > u_c$ with higher values for smaller values of λ in the numerical experiments. We choose the value of parameter $k = 1$ for lower solution and $k \geq 1$ for upper solutions (multiplicity of solutions). For the numerical results, the residual norm $||r||$ is defined, so that the scaling is independent of the grid size h:

$$||r|| = \sqrt{\frac{\sum_{i,j} r_{i,j}^2}{J}}, \tag{3.1}$$

and the nonlinear stopping criterium reads: $\|r\| \leq tol = 10^{-10}$. The tolerance of the linear multigrid method within the Newton iterations is set to be 10^{-8}. The following numerical experiments illustrate the convergence of all types of numerical solutions (lower, upper, periodic and semi-periodic) and demonstrate the effectiveness of the proposed numerical methods for all parameter values $\lambda \in (0, \lambda_c]$. Furthermore, we examine the numerical bifurcation behaviour of the two-dimensional Bratu problem by approximating the nonlinear term and identify the existence of multiple solutions. Numerical results show the comparison of the performance of the three numerical strategies for small values of λ on an increasing number of grid points.

3.1. Experiment 1: the two-dimensional Bratu problem

Consider the classical two-dimensional Bratu model with homogeneous Dirichlet boundary conditions on the domain $\Omega = [0, 1]^2$:

$$\begin{aligned} &\Delta u + \lambda\, \mathrm{e}^u = 0, \qquad (x, y) \in \Omega, \\ &u|_{\partial\Omega} = 0, \end{aligned} \tag{3.2}$$

where $u = u(x, y)$ and $\lambda \geq 0$. Unlike the situation in the one-dimensional problem [30], *no* exact solution $u(x, y)$ is known for (3.2). Note that the solution, in the two-dimensional case depends heavily on the parameter λ. The Bratu problem (3.2) has no, one or two solutions, leading to two branches (lower solution and upper solution), for $\lambda > \lambda_c$, $\lambda = \lambda_c$ or $\lambda < \lambda_c$ respectively as described in Fig. 1. The value of the turning point $\lambda_c \approx 6.808124423$ can be calculated numerically by using a Newton method (also reported in [8,19]). The solutions of model (3.2) attain their maximum at, and are point-symmetric around, $(\frac{1}{2}, \frac{1}{2})$.

In earlier numerical approaches [8,10,11,13], researchers reported that the convergence to the lower solution can easily be obtained for all values of $0 < \lambda \leq \lambda_c$. Table 1 shows the performance of three proposed numerical methods to obtain the lower solution of problem (3.2) for $\lambda = 1$ on increasing grid points. Results indicate that the three described methods

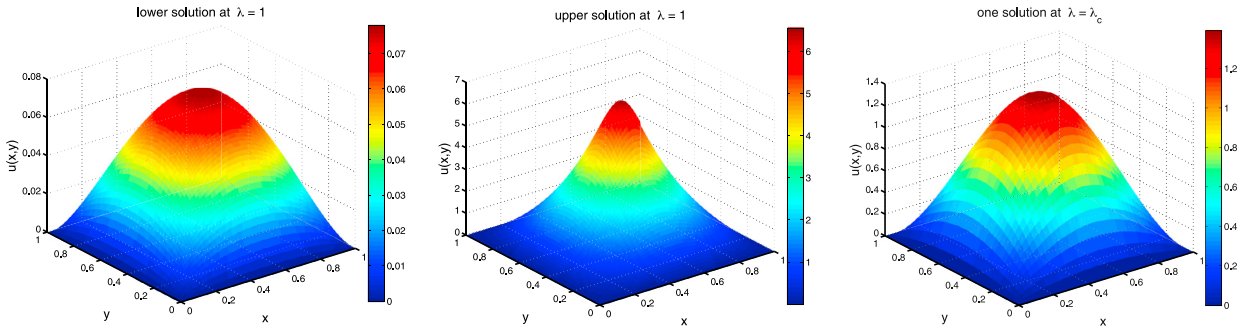


Fig. 1. two solutions for $\lambda = 1$ (left and middle panel) and the unique solution for $\lambda = \lambda_c$ (right panel) of the Bratu-model (3.2) using FAS-MG on 161 × 161 grid points.

Table 1
the number of iterations and the CPU time in seconds for the lower solution of model (3.2) for $\lambda = 1$ and for an increasing number of grid points.

Grids	FD-Method		Newton-MG		FAS	
J	Itr	Time	Newton-Itr (linear-Itr)	Time	V-cycles	Time
11^2	5	0.8846	4 (4,5,3,1)	0.0246	5	0.0100
21^2	6	2.7255	4(6,8,4,3)	0.0524	5	0.0326
41^2	6	9.7324	5(5,8,6,5,4)	0.2188	5	0.0818
81^2	6	23.0232	5(6,6,4,3,3)	0.6784	6	0.2460
161^2	8	196.7221	5(6,6,5,5,6)	7.2526	6	1.3182
321^2	8	266.3218	5(8,6,6,5,6)	15.5473	5	3.9394

Table 2
the number of iterations and the CPU time in seconds for the upper solution of model (3.2) for $\lambda = 1$ and for an increasing number of grid points.

Grids	FD-Method		Newton-MG		FAS	
J	Itr.	Time	Newton-Itr (linear-Itr)	Time	V-cycles	Time
11^2	11	2.3422	8 (12,9,6,25,8,6,1,1)	2.5606	32	0.2764
21^2	12	6.0568	8 (23,12,28,7,4,5,2,2)	5.3318	46	0.7564
41^2	11	8.7054	11 (32,18,22,56,14,12,25,12,9,3,2)	8.3275	35	0.9437
81^2	14	41.1728	11 (42,63,56,24,36,24,46,11,6,1,1)	16.1121	32	1.3693
161^2	28	448.0945	9 (49,84,66,18,6,4,1,1,1)	27.9396	32	6.9758
321^2	48	926.4628	11 (33,68,56,32,21,15,13,11,5,3,3)	41.8759	42	11.5745

successfully obtain the convergence of lower solution. However, FAS-MG and Newton-MG methods are more efficient than FD-method with respect to time.

Numerical difficulties occur when the second branch of the solution (the upper solution) needs to be computed particularly for small values of λ. In such a case, numerical methods are very sensitive with respect to the initial guess, especially for small values of λ and when λ approaches λ_c. Table 2 shows the comparison of three methods to get the upper solution of model (3.2) for $\lambda = 1$ for decreasing grid size h. Numerical results indicate that the convergence of the upper solution slows down with the increase of the number of grid points in FD-methods. Newton-MG with *nested* iteration needs to calculate the Jacobian matrix at each iteration. The convergence process also slows down when we increase the size of problem. In FAS-MG, we use a $V(2, 2)$ cycle with MINRES as a relaxation smoother (both for pre and post relaxation).

The difference between the performance of methods is more pronounced when we try to find the upper solution for small values of λ. Fig. 2 shows the upper solution of model (3.2) obtained when we employ the FAS-MG method for small values of λ on a grid of 161 × 161 grid points.

For small values of λ, Table 3 demonstrates the convergence to the upper solution and provides a comparison of the efficiency of the three methods for 321 × 321 grid points. It is observed that for small grid sizes and especially for small values of λ, the convergence of the upper solutions becomes very slow and this affects the efficiency for both Newton-MG and FD methods. Particularly, for FAS-MG the convergence is slower, but the rate of slowing down is not very high. This shows the effectiveness of the FAS-MG method over the other two methods.

The convergence of the numerical solutions, especially for the upper solution, is of great importance. In earlier numerical reports, it has always been hard, or even impossible, to find convergence to the second branch in the 2D Bratu problem (3.2) for $h \downarrow 0$. We investigated numerically the convergence behaviour for the upper solution when applying the three described methods. Fig. 3 illustrates nicely the numerical convergence for nonlinear BVP (3.2) by using the proposed numerical

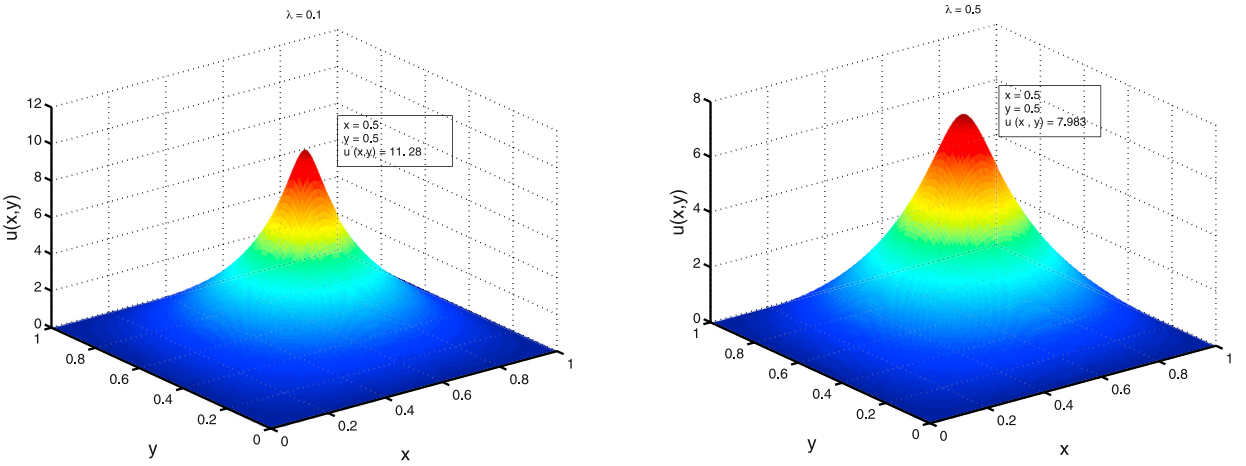
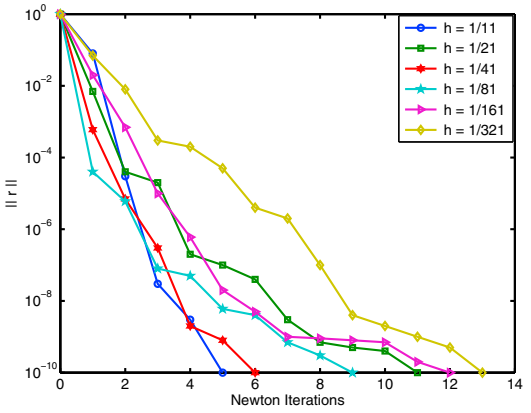
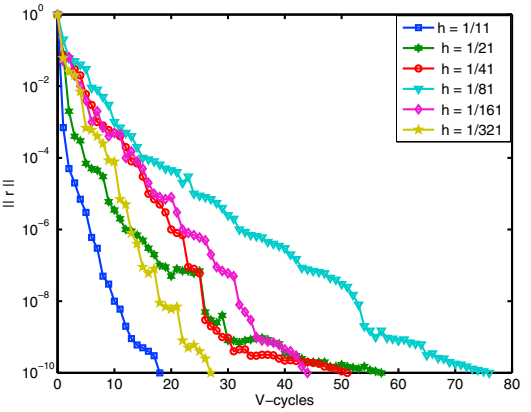


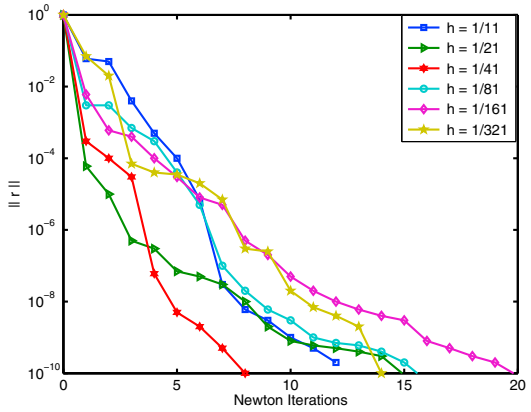
Fig. 2. the upper solution of model (3.2) for small values of λ on 161 × 161 grid points. For λ = 0.1 (left) and λ = 0.5 (right): FAS-MG-V(2,2) was employed to obtain the upper solutions.



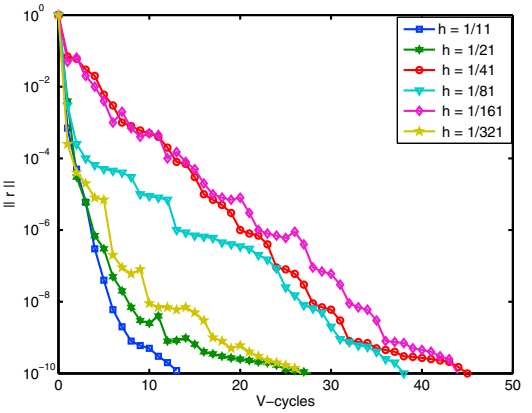
Convergence for nested Newton-MG for $\lambda = 0.1$



Convergence for FAS-MG for $\lambda = 0.1$



Convergence for nested Newton-MG for $\lambda = \lambda_c$



Convergence for FAS-MG for $\lambda = \lambda_c$

Fig. 3. the numerical convergence for the upper solution for λ = 0.1 and the unique solution for $\lambda = \lambda_c$ of problem (3.2) on decreasing grid sizes *h* using the nested Newton-MG method (left two figures) and the FAS-MG-V(2, 2) method (right two figures).

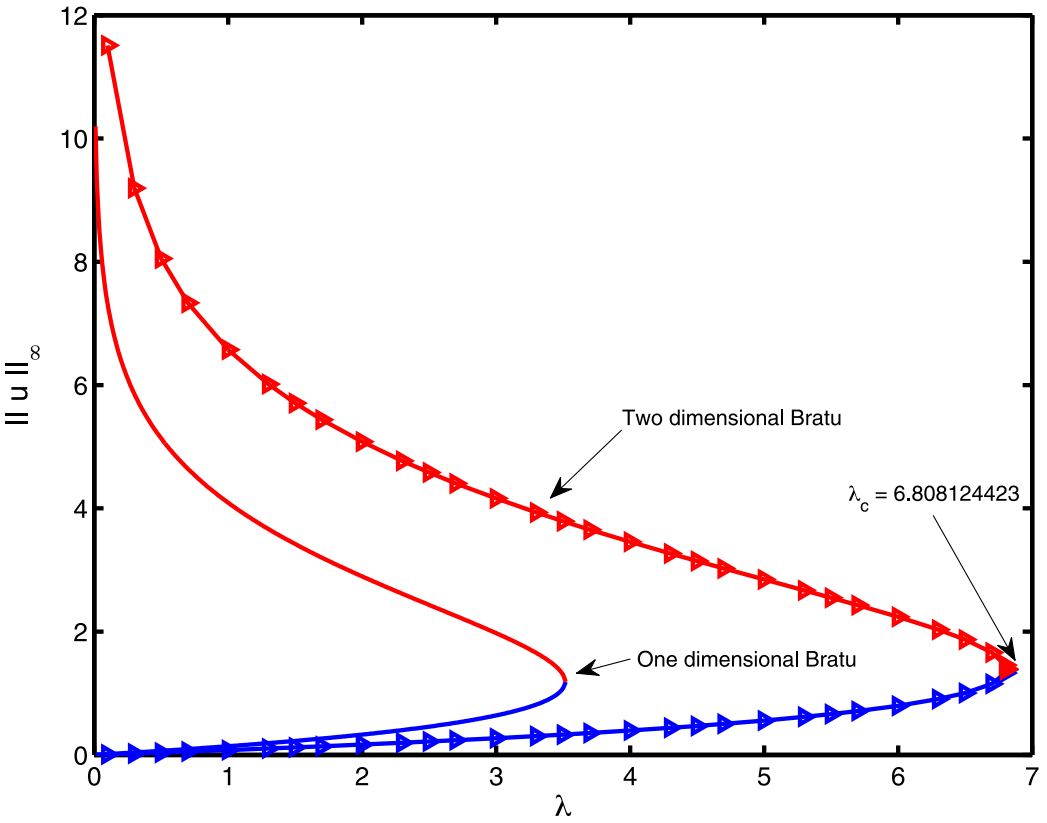


Fig. 4. the bifurcation diagram of the two-dimensional Bratu-model (3.2) using the FAS-MG-MINRES approach on 161 × 161 grid points, compared to the 1D case (see also [30]). The critical value $\lambda_c \approx 6.808124423$.

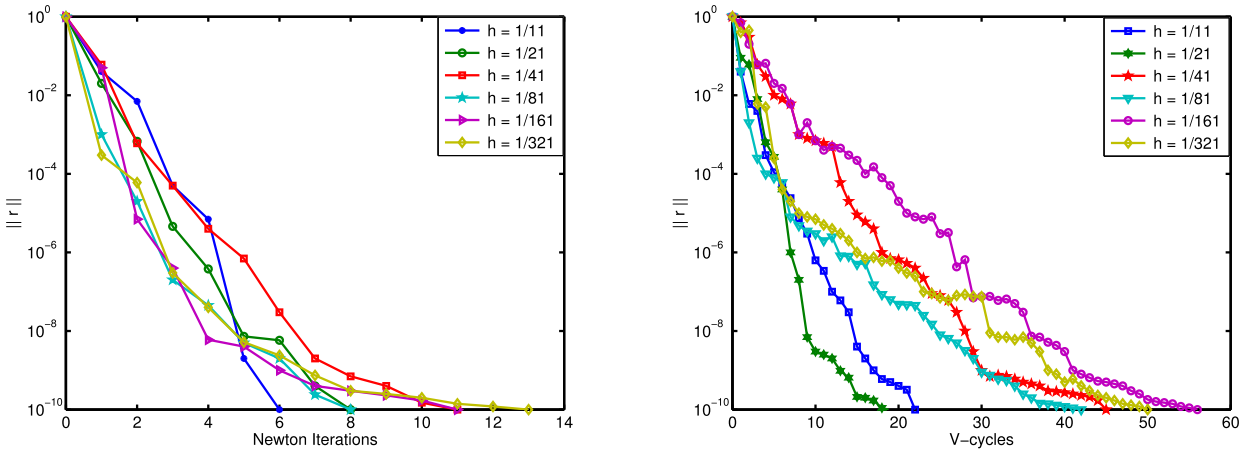


Fig. 5. numerical convergence for the unique solution of model (3.3) for $\lambda = \lambda_c$ using nested Newton-MG (left) and FAS-MG-V(2, 2) (right) for decreasing values of the grid sizes.

Table 3
the number of iterations and the CPU time in seconds for the upper solution of model (3.2) for small values of λ on 321 × 321 grid points.

	FD-Method		Newton-MG		FAS	
λ	Itr	Time	Newton-Itr (linear-Itr)	Time	V-cycles	Time
0.1	63	1123. 3865	13 (74,46,22,48,92,76,18,36,42,12,8,5,5)	211.9268	46	84.7324
0.3	48	838.7255	11 (85,33,76,36,62,25,12,28,8,12,1)	152.1488	91	41.2247
0.5	52	774.1730	9 (42,38,82,25,62,34,15,8,6)	112.5773	42	18.8102
λ_c	12	376.6732	12 (50,23,38,32,18,12,15,7,5,3,3,2)	42.7634	34	8.2155

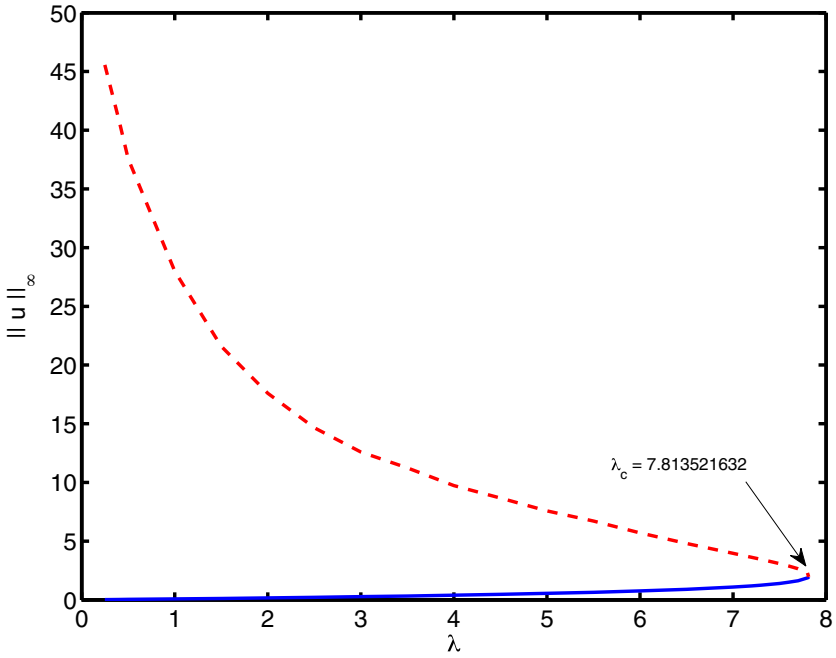


Fig. 6. the bifurcation curve for problem (3.3) is obtained by using FAS-MG on 161 × 161 grid points with the critical value $\lambda_c \approx 7.813521632$. The solid line (blue) denotes the lower solution and the dashed line (red) the upper solution, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

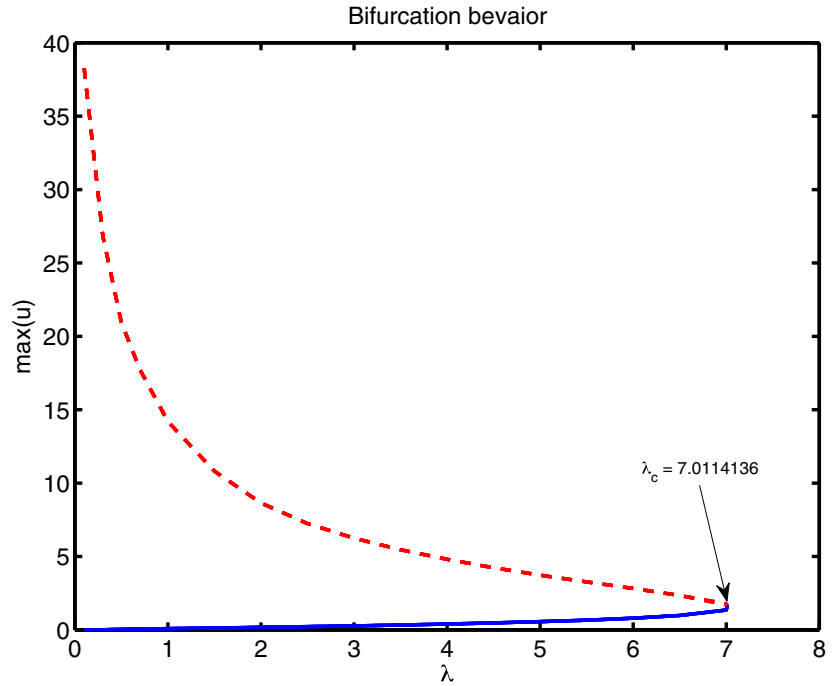


Fig. 7. the bifurcation curve for model (3.4) is obtained by employing FAS-MG (for positive solutions only) for different values of λ. Red (dashed line) and blue (solid line) show the upper and lower solutions, respectively. For model (3.4) the critical value $\lambda_c \approx 7.011421641$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

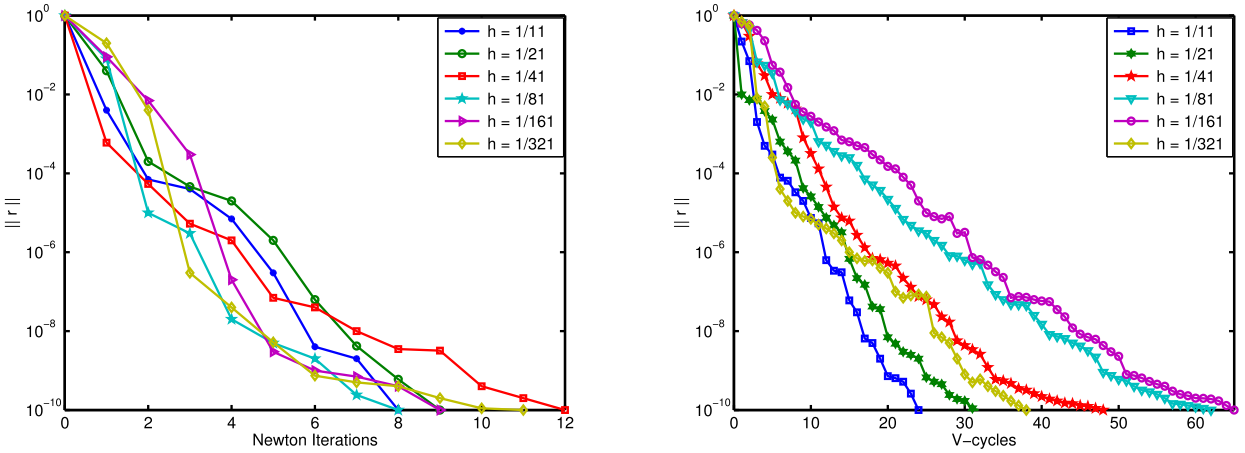


Fig. 8. numerical convergence of Newton-MG (left) and FAS-MG (right) for the unique solution at different grid sizes (model (3.4) for $\lambda = \lambda_c$).

Table 4
the number of iterations and the CPU time in seconds for the lower solution of model (3.3) for small values of λ on 321 $\times$ 321 grid points.

	FD-Method		Newton-MG		FAS	
λ	Itr	Time	Newton-Itr	Time	V-cycles	Time
0.1	13	196.7281	6 (16,22,6,15,3,3)	103.04831	12	34.7122
0.2	11	166.2892	5 (12,18,8,4,4)	78.2824	8	21.9314
0.3	11	142.7429	6 (22,15,18,8,2,5)	62.2669	10	19.0612
0.5	15	129.2432	6 (8,6,12,6,3,3)	40.8436	8	15.0946
1	13	111.8652	5 (8,13,5,5,4)	15.7763	8	6.1644

approach. The numerical experiments demonstrate that the proposed methods converge rapidly, for both the lower and the upper solutions. These results yield better accuracy and are more efficient than previously reported numerical schemes, showing the effectiveness of proposed method.

Numerical results clearly indicate that FAS-MG is more efficient than the Newton-MG and the FD method for computing both positive solutions (lower and upper) of BVP (3.2) for $\lambda \in (0, \lambda_c]$. The numerical bifurcation diagram is presented in Fig. 4 of the two-dimensional Bratu model (3.2), using the FAS-MG on a grid of 161 $\times$ 161 grid points. The upper and lower solutions are presented by red and blue lines, respectively. It is observed that both one-dimensional and two-dimensional GB models depict similar bifurcation behaviour.

It has been reported in [30] for the one-dimensional (1D) Bratu model case, that new types of solutions (periodic and semi-periodic) exist by considering a Taylor approximation of the nonlinear term e^u. Motivated by the 1D case in [30], we extend the numerical results for the 2D Bratu problem. For this, we take the Taylor approximation of e^u up to quadratic and cubic terms of problem (3.2) and present new types of solutions. Bifurcation diagrams are presented in the experiments 2 and 3 in the next sections.

3.2. Experiment 2: a quadratic approximation

We consider a Taylor approximation of e^u up to the quadratic term. Then model (3.2) with Dirichlet boundary conditions can be written as:

$$\begin{aligned} &\Delta u + \lambda\,(1 + u + \tfrac{1}{2}u^2) = 0, \qquad (x, y) \in [0, 1]^2, \\ &u|_{\partial\Omega} = 0. \end{aligned} \tag{3.3}$$

The solution behaviour of model (3.3) resembles that of the classical Bratu problem (3.2) in the sense that two, one or zero solutions exist for $\lambda < \lambda_c$, $\lambda = \lambda_c$ or $\lambda > \lambda_c$, respectively. For this case the critical value $\lambda_c \approx 7.813521632$ is computed by using Newton's method. Tables 4 and 5 summarize the performance of described methods when employed to obtain the solutions for model (3.3). It can be observed that for small values of λ, the proposed methods converge to both solutions (lower and upper) successfully on a grid of 321 $\times$ 321 grid points. Numerical results indicate that the performance of the proposed methods for model (3.3) is better than for problem (3.2) as $f(u)$ is not a highly nonlinear function in model (3.3). Further it is noted that FAS-MG is found to be more efficient than nested Newton-MG and the FD-method with respect to computational time.

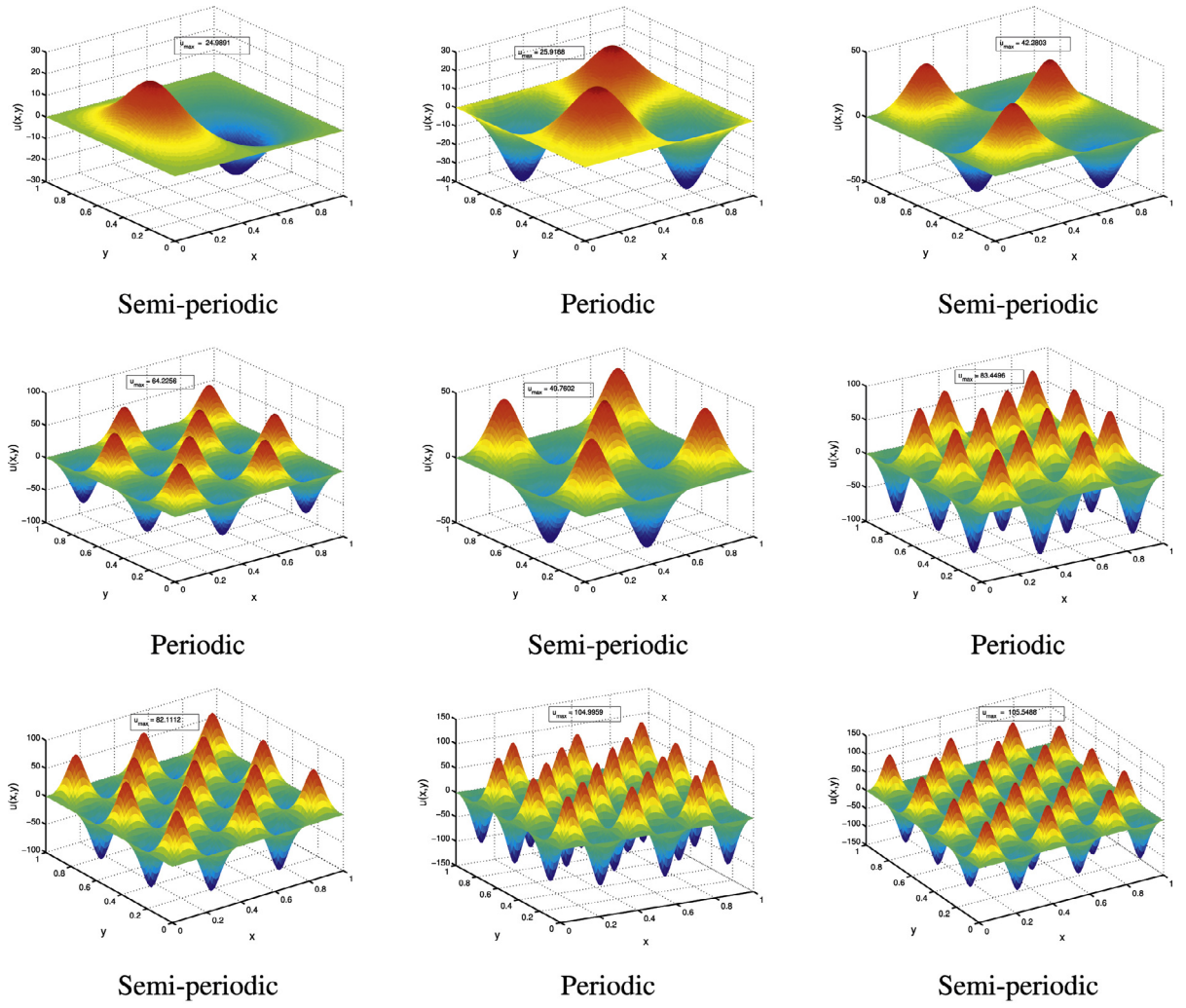


Fig. 9. some periodic and semi-periodic solutions for $\lambda = 1$ of model (3.4) obtained by using FAS-MG with 101×101 grid points.

Table 5
the number of iterations and the CPU time in seconds for the upper solution of problem (3.3) for small values of λ on 321×321 grid points.

	FD-Method		Newton-MG		FAS	
λ	Itr	Time	Newton-Itr (linear-Itr)	Time	V-cycles	Time
0.1	62	867 0.8210	11(72,96,65,84,58,37,18,5,7,9,3)	148.3319	92	39.7621
0.2	60	834.2298	9 (92,56,52,88,44,15,9,5,6)	124.0352	68	34.2986
0.3	66	844.3205	8 (46,82,96,38,42,18,5,1)	96.3809	84	29.3153
0.5	45	633.9921	12 (65,54,90,36,28,15,24,15,9,5,7,5)	74.3112	48	15.0037
1	48	624.3172	8 (48,67,32,16,9,12,5,1)	42.4507	42	9.1496

In addition to the above mentioned solutions, we also provide the numerical convergence for the unique solutions of model (3.3) for the critical value λ_c. The convergence to the unique solution is obtained by using both Newton-MG and FAS-MG methods for decreasing values of h, see Fig. 5.

Furthermore, we investigate the numerical bifurcation behaviour of model (3.3) for different values of $\lambda \in (0, \lambda_c]$. The numerical bifurcation diagram is presented in Fig. 6, using the FAS-MG on 161×161 grid points. The upper and lower solutions are presented by red (dash) and blue (solid) curves respectively. It is observed that despite having different critical values for two-dimensional nonlinear models (3.2) and (3.3), both depict the same bifurcation behaviour (see Figs. 4 and 6).

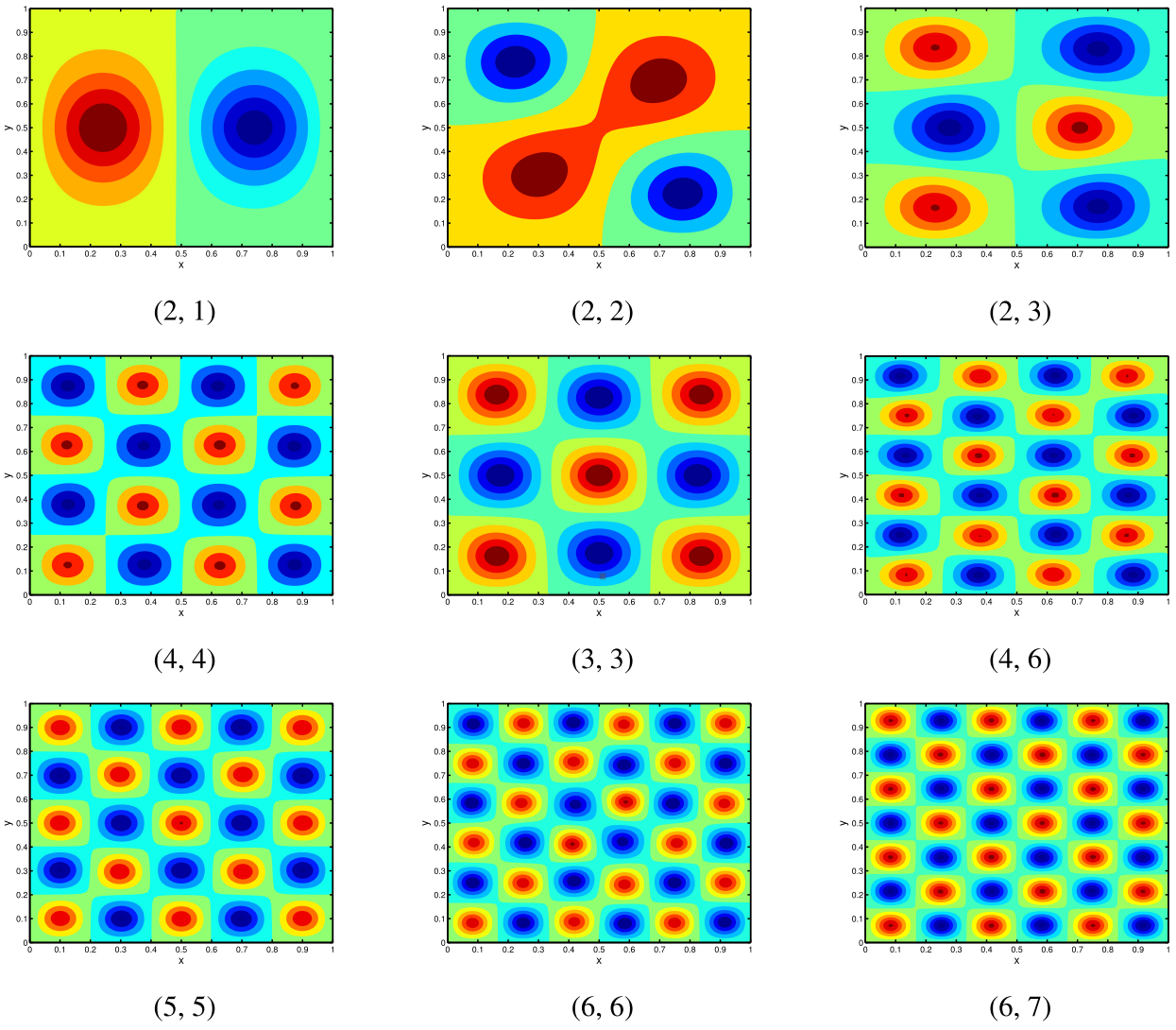


Fig. 10. the corresponding contour plots of periodic and semi-periodic solutions of Fig. 9 for $\lambda = 1$ of model (3.4).

Moreover, like in the 1D case [30], a similar bifurcation behaviour can be observed by approximating the nonlinear term e^u up to even-power approximations.

3.3. Experiment 3: a cubic approximation

Next, we approximate the nonlinear term of e^u up to the cubic term:

$$\begin{aligned} &\Delta u + \lambda\,(1 + u + \tfrac{1}{2}u^2 + \tfrac{1}{6}u^3) = 0, \qquad (x, y) \in \Omega = [0, 1]^2, \\ &u|_{\partial\Omega} = 0. \end{aligned} \tag{3.4}$$

This model also posseses the same properties as described for models (3.2) and (3.3), when considering positive solutions only. We computed the critical value $\lambda_c \approx 7.011421641$ by using Newton's method. Numerical results for model (3.4) are presented in Tables 6 and 7 for different small values of λ on a 321×321 grid. Numerical convergence for the critical value λ_c is shown in Fig. 8. As for experiment 2, it is again observed that FAS-MG is more efficient than the nested Newton-MG method and the FD-method with respect to computational time. These results demonstrate the validity and applicability of the FAS-MG method as presented in Section 2.

The numerical bifurcation curve for positive solutions, shows two solutions (lower and upper) and a unique solution using the FAS-MG method for $\lambda \in (0, \lambda_c]$ (see Fig. 7).

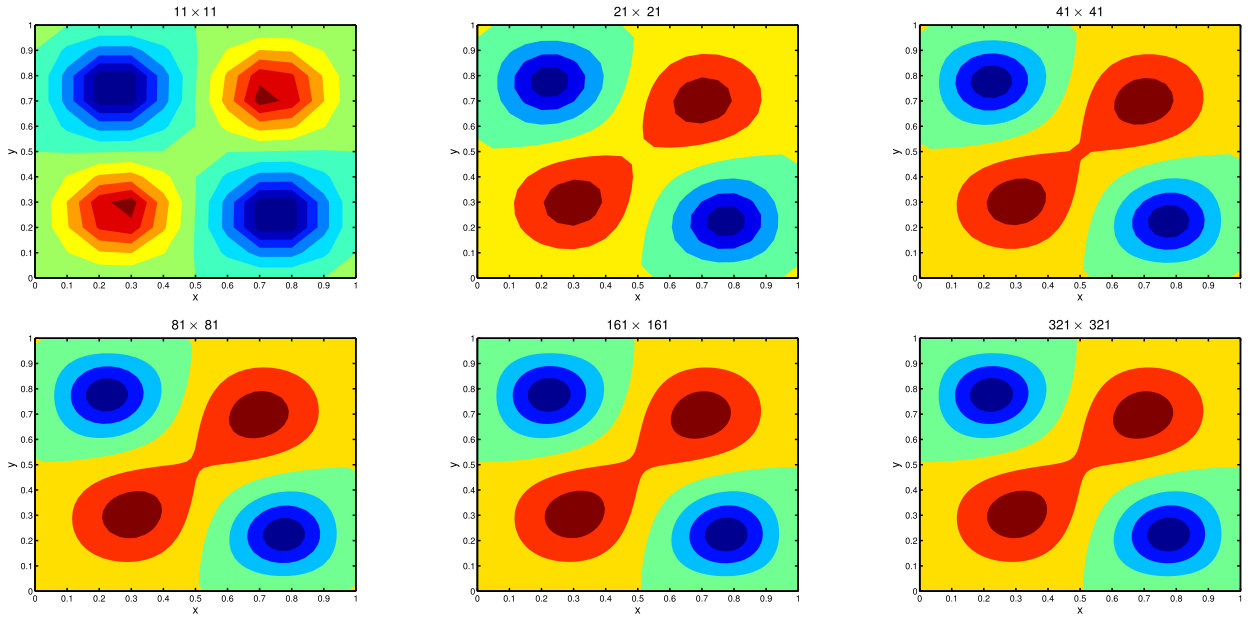


Fig. 11. contour plots of the numerical convergence in the case of the (2, 2) periodic solution on an increasing number of grid points for $\lambda = 1$ in model (3.4). The results are obtained by using the FAS-MG-V(2, 2) method.

Table 6
the number of iterations and the CPU time in seconds for the 'lowest' solution in problem (3.4) for small values of λ.

	FD-Method		Newton-MG		FAS	
λ	Itr	Time	Newton -Itr (linear Itr)	Time	V-cycles	Time
0.1	15	214.8177	8 (22,18,12,15,6,4,8,3)	116.2213	12	42.1348
0.2	13	171.0430	6 (18,26,22,14,5,7)	94.3319	14	22.8256
0.3	11	146.2219	8 (15,22,18,11,9,5,5,1)	66.1428	12	19.8403
0.5	11	134.2301	5 (11,15,9,5,2)	43.9287	12	15.2298
1	9	118.0972	6 (11,9,8,5,1,1)	18.0525	8	6.6635

Table 7
the number of iterations and the CPU time in seconds for the upper solution of problem (3.4) for small values of λ.

	FD-Method		Newton-MG		FAS	
λ	Itr	Time	Newton-Itr (linear Itr)	Time	V-cycles	Time
0.1	72	894.1275	13 (56,62,48,35,33,30,44,20,15,11,11,9,2)	151.7743	86	42.2361
0.2	56	826.3816	11 (66,75,18,22,26,36,9,5,3,3,1)	126.2758	68	37.1184
0.3	68	719.0528	11 (46,36,22,20,34,18,12,11,5,7,3)	108.0543	84	35.2846
0.5	56	664.1194	12 (36,42,35,18,12,9,9,9,12,5,1,1)	81.3286	48	21.8137
1	44	641.0037	11 (46,54,22,34,12,12,8,6,3,7,5)	49.3371	36	14.7414

3.3.1. Periodic and semi-periodic solutions

In our previous article, [30], we found that, for this cubic approximation, more than two solutions may exist. These solutions can be either periodic or semi-periodic and the number of solutions varies from 0, 1, 2 up to ∞. We define these periodic and semi-periodic solutions in the following way:

$$(l, m) = \begin{cases} Periodic & ; \quad l \,\&\, m = 2, 4, 6, 8, \ldots, \\ Semi - periodic & ; \quad otherwise \end{cases} \tag{3.5}$$

where l and m are the number of extrema along the horizontal and vertical straight lines respectively in solution $u(x, y)$. For example, (2, 3) represents a semi-periodic solution, referring to the definition in (3.5). It indicates that the solution $u(x, y)$ has $l = 3$ extrema along the horizontal axis and $m = 2$ extrema along the vertical axis. Similarly, (3,3) represents a semi-periodic solution, whereas (4,2) and (6,6) represent a periodic solution as l and m are even numbers. To find this new multiplicity of solutions, we employed the FAS-MG method which is found as more efficient than the other two methods

(see experiment 1 and 2). Both the periodic and semi-periodic solutions are obtained successfully with the $V(2, 2)$-cycle in FAS-MG for $\lambda = 1$ on 161×161 grid points. Some of the periodic and semi-periodic solutions and their corresponding contour plots for $\lambda = 1$ are presented in Fig. 9 and Fig. 10, respectively.

Furthermore, it is important to note that the numerical convergence for these solutions is achieved successfully. The numerical convergence for the case (2,2) periodic solution and for $\lambda = 1$ as a function of the grid size h is depicted in Fig. 11. These results again confirm the effectiveness of the FAS-MG method. However, for decreasing grid sizes, FAS-MG could not obtain the desired accuracy. Moreover, a similar bifurcation behaviour (for positive solutions only) can be obtained by approximating the nonlinear term e^u up to odd power approximations like in the 1D case [30].

The three numerical experiments as discussed above, clearly show that the proposed methods, namely, a FD based method, Newton-MG and FAS-MG converge successfully for the two-dimensional GB model for small values of λ on an increasing number of grid points. However, the efficiency in terms of computational time of FAS-MG is better than for the other two methods. Furthermore, FAS-MG is used successfully to obtain the convergence of new multiple solutions with greater accuracy and efficiency. Both FD and Newton-MG methods lose efficiency and desired accuracy for decreasing grid sizes when trying to compute new multiple solutions. This shows the effectiveness of FAS-MG.

4. Conclusions

In this paper, we have numerically computed the two-dimensional Bratu-type nonlinear boundary-value problems. We proposed three numerical methods, viz, a finite-difference based method with *fsolve* and the two MG approaches: Newton-MG and FAS-MG with the MINRES method as relaxation smoother. With these methods we have been able to overcome the convergence difficulties without compromising the efficiency. A comparison, in terms of convergence, accuracy and efficiency between the three numerical methods demonstrated an improvement for the whole parameter range $\lambda \in (0, \lambda_c]$. Furthermore, we investigated the bifurcation behaviour of solutions for the two-dimensional Bratu models and found new multiplicity of solutions in the case of cubic approximation of the nonlinear exponential term. It has been demonstrated successfully that the convergence of all solutions namely, unique, lower, upper, periodic and semi-periodic has been obtained for small values of the parameter λ. FAS-MG is found to be more efficient than the other two methods as observed in numerical results. This success is achieved due to an appropriate initial guess and with the use of the MINRES method as relaxation smoother.

Acknowledgments

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