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Method of lines study of nonlinear dispersive waves

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Abstract

In this study, we consider partial differential equation problems describing nonlinear wave phenomena, e.g., a fully nonlinear third order Korteweg-de Vries (KdV) equation, the fourth order Boussinesq equation, the fifth order Kaup–Kupershmidt equation and an extended KdV5 equation. First, we develop a method of lines solution strategy, using an adaptive mesh refinement algorithm based on the equidistribution principle and spatial regularization techniques. On the resulting highly nonuniform spatial grids, the computation of high-order derivative terms appears particularly delicate and we focus attention on the selection of appropriate approximation techniques. Finally, we solve several illustrative problems and compare our computational approach to conventional solution techniques.

Keywords: Method of lines; Adaptive mesh refinement; Finite differences; N-soliton solution; Korteweg-de Vries equation; Kaup–Kupershmidt equation

1. Introduction

In recent years, much interest has developed in the numerical treatment of partial differential equations (PDEs) describing nonlinear wave phenomena, and particularly solitary waves. In this study, we consider PDEs with high-order spatial derivatives, e.g. a fully nonlinear Korteweg-de Vries (KdV)-like equation (featuring compactons), the "good" Boussinesq equation, the Kaup–Kupershmidt (KK) equation and an extended KdV5 equation. These equations are used to model nonlinear

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dispersive waves in a wide range of application areas, such as water wave models, laser optics, and plasma physics.

In order to efficiently compute numerical solutions to these equations and to accurately resolve sharp spatial variations in the solution curves, we use an adaptive grid technique that automatically concentrates the spatial grid points in the regions of high solution activity (see, e.g. [8], for a presentation of several adaptive grid solution techniques). Here, we develop a method of lines (MOL) solution strategy based on an adaptive mesh refinement (AMR) algorithm.

2. Adaptive MOL solution

In this section, an AMR algorithm which equidistributes a given monitor function subject to constraints on the grid regularity is presented. The time-stepping procedure as well as some implementation issues are discussed.

2.1. Grid equidistribution with constraints

Consider the PDE problem

$$u_t = f(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx}), \quad x_L < x < x_R, \tag{1}$$

where u is the vector of dependent variables, and the subscript notation denotes partial derivatives, e.g. $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$.

In order to solve this problem numerically, a spatial grid is built so as to equidistribute a specified monitor function $m(u)$. The spatial equidistribution equation for the grid points x_i, $i = 1, 2, \ldots, N$, can be expressed in continuous form

$$\int_{x_{i-1}}^{x_i} m(u)\,\mathrm{d}x = \int_{x_i}^{x_{i+1}} m(u)\,\mathrm{d}x = c, \quad 2 \leqslant i \leqslant N-1, \tag{2}$$

or in discrete form

$$M_{i-1}\Delta x_{i-1} = M_i \Delta x_i = c, \quad 2 \leqslant i \leqslant N-1, \tag{3}$$

where $\Delta x_i = x_{i+1} - x_i$ is the local grid spacing, M_i is a discrete approximation of the monitor function $m(u)$ in the grid interval $[x_i, x_{i+1}]$, and c is a constant.

A popular monitor function is based on the arc length of the solution [7], i.e.

$$m(u) = \sqrt{\alpha + ||u_x||_2^2}. \tag{4}$$

In this expression, $\alpha > 0$ ensures that the monitor function is strictly positive and acts as a regularization parameter which forces the existence of at least a few nodes in flat parts of the solution.

The accuracy of the spatial derivative approximations (e.g., using finite differences) and the stiffness of the semi-discrete system of differential equations are largely influenced by the regularity and spacing of the grid points. This stresses the importance of limiting grid distortion using spatial regularization procedures.

Here, a procedure due to Kautsky and Nichols [5] based on the concept of a *locally bounded grid* is used. This procedure, as we shall see, involves a *variable number of nodes*. A grid is said to be locally bounded with respect to a constraint $K \geqslant 1$ if

$$\frac{1}{K} \leqslant \frac{\Delta x_i}{\Delta x_{i-1}} \leqslant K, \quad 2 \leqslant i \leqslant N-1. \tag{5}$$

Then the equidistribution problem becomes: Given a monitor function $m \in C^+$ (the set of continuous piecewise functions on $[x_L, x_R]$) and constants $c > 0$ and $K \geqslant 1$, find the grid which is

(1) sub-equidistributing with respect to m and c on $[x_L, x_R]$, i.e., for the smallest number of nodes N such that $Nc \geqslant \int_{x_L}^{x_R} m\,\mathrm{d}x$, we have $\int_{x_i}^{x_{i+1}} m\,\mathrm{d}x \leqslant c$;
(2) locally bounded with respect to K.

The idea of the solution to this problem, which is developed in [5], is to increase the given monitor function m—in a procedure which is called "padding"—in such a way that, when the padded monitor function is equidistributed, the ratio of consecutive grid steps is bounded as required. The padding is chosen so that the equidistributing grid has adjacent steps with constant ratios equal to the maximum allowed. Such a function exists and is given by the following formal results [5]:

Let λ be a given number. For any $m \in C^+$, we define a padding $P(m)$ of m by

$$P(m)(z) = \max_{x \in [x_L, x_R]} \frac{m(x)}{1 + \lambda |z - x| m(x)}. \tag{6}$$

$P(m)$ has the properties:

(1) $P(m)$ is strictly positive on $[x_L, x_R]$, except in the case $m \equiv 0$;
(2) $P(m) \geqslant m$ on $[x_L, x_R]$;
(3) $P(P(m)) = P(m)$ on $[x_L, x_R]$.

Let $\lambda > 0$, $m \in C^+$ and a grid π be equidistributing on $[x_L, x_R]$ with respect to $P(m)$ and some $c > 0$. Then

(1) the grid π is sub-equidistributing with respect to m and c;
(2) for $K = e^{\lambda c}$ we have

$$\frac{1}{K} \leqslant \frac{\Delta x_i}{\Delta x_{i-1}} \leqslant K, \quad i = 2, \ldots, N-1.$$

Based on these results, it is now possible to build a grid which is sub-equidistributing with respect to m and c and which is locally bounded with respect to K. In practice, the algorithm proceeds as follows:

(1) pad the monitor function using $\lambda = (\log K)/c$;
(2) determine the smallest number of nodes N such that $Nc \geqslant \int_{x_L}^{x_R} P(m)\,\mathrm{d}x$;
(3) equidistribute $P(m)$ with respect to $d = (\int_{x_L}^{x_R} P(m)\,\mathrm{d}x)/N$.

As $d \leqslant c$, the grid is locally bounded with respect to a constant $L \leqslant K$, so that the number of points in the grid may be greater than required to strictly satisfy the problem constraints.

2.2. Time-stepping procedure and implementation details

The AMR is a static procedure and as such, proceeds in four separate steps:

(1) approximation of the spatial derivatives on a fixed nonuniform grid;
(2) time integration of the resulting semi-discrete ODEs;
(3) adaptation/refinement of the spatial grid;
(4) interpolation of the solution to produce new initial conditions.

In step (1), the spatial derivatives are approximated using finite difference approximations up to any level of accuracy on a nonuniform grid as implemented in the standard Fortran subroutine WEIGHTS by Fornberg [2]. This algorithm is used for generating "direct" as well as "stagewise" schemes. In the latter case, higher-order derivatives are obtained by successive numerical differentiations of lower-order derivatives.

In step (2), time integration of the semi-discrete system of stiff ODEs or DAEs is accomplished using the variable step, fifth-order, implicit Runge–Kutta solver RADAU5 [3]. Time integration is halted periodically, i.e. every N_{adapt} integration steps, to adapt/refine the spatial grid.

In step (3), the grid is updated using the algorithm described in the previous section. Implementation issues involves computation of the monitor function (4) using cubic spline differentiators, padding of the monitor function in two sweeps of the grid (in the forward and backward direction), grid equidistribution by inverse linear interpolation based on a trapezoidal rule.

Finally, in step (4), the solution is interpolated using cubic splines in order to generate initial conditions on the new grid.

3. Case studies

3.1. An extended nonlinear KdV equation

While seeking to understand the role of nonlinear dispersion in the formation of patterns in liquid drops, Rosenau and Hyman [6] have introduced a family of fully nonlinear KdV-like equations in the form

$$u_t + (u^m)_{xx} + (u^n)_{xxx} = 0, \quad m > 0, \; 1 < n \leqslant 3. \tag{7}$$

These equations, which are denoted $K(m,n)$, have the property that for certain m and n, their solitary wave solutions have compact support. This remarkable property suggested the name "compacton" to these authors. Particularly, the $K(2,2)$ equation possesses a solitary wave solution with a compact support given by

$$\begin{aligned} u_{\mathrm{c}}(x,t) &= \frac{4s}{3}\cos^2\left(\frac{x-st}{4}\right), \quad |x-st| \leqslant 2\pi, \\ &= 0 \quad \text{otherwise}, \end{aligned} \tag{8}$$

where s is the compacton velocity. Although the second derivative of the compacton is discontinuous at its edges, (8) is a strong solution of Eq. (7) since the third derivative is applied to u^2, which has three smooth derivatives everywhere including the edge.

Table 1
Propagation of a single compacton: computational statistics and average values of the L_2-norm of the error

Grid	N	Approx.	N_{adapt}	FNS	JACS	STEPS	CPU(s)	$\|e\|_2$
Adaptive	203–204	$((u_x)_x)_x$	1	1813	228	448	6.9	$\approx 1.5 \cdot 10^{-3}$
Adaptive	203–204	u_{xxx}	1	2922	472	716	12.8	$\approx 3.8 \cdot 10^{-4}$
Uniform	501	$((u_x)_x)_x$	—	2617	220	241	15.8	$\approx 4.5 \cdot 10^{-3}$

We consider the propagation of a single compacton with speed $s=0.5$ over the time interval $(0, 80)$. Accordingly, homogeneous Dirichlet boundary conditions are imposed in $x_L = -30$ and $x_R = 70$. As stressed in [6], there are several numerical difficulties in solving the $K(2, 2)$ equation, which are due to nonlinear dispersion and the lack of smoothness at the edge of the compacton, possibly leading to instability.

Following the MOL, it was not possible to solve satisfactorily the $K(2, 2)$ equation on a fixed uniform grid, with the exception of the particular setting:

- stagewise differentiation of the nonlinear dispersive term, i.e. $(u^2)_{xxx} = (((u^2)_x)_x)_x$, using a 3-point centered finite difference scheme (for computing a first derivative);
- $N = 501$ grid points;
- time integration with $atol = rtol = 10^{-5}$.

Even in this fortuitous situation, the graph of the solution displays unacceptable downstream oscillations. However, any attempt to improve on this situation by increasing the number of nodes or reducing the error tolerances lead to failure of the simulation run. The use of a (direct) 7-point centered finite difference scheme for computing $(u^2)_{xxx}$ was also unsuccessful.

When using AMR (rather than a fixed grid method), both direct and stagewise finite difference schemes can be used to solve the problem on the time interval $(0, 80)$. The tuning parameters take the following values: $\alpha = 10^{-6}$, $c = 0.01$, $K = 1.1$ and $N_{adapt} = 1$ (the grid is adapted after each time step). The corresponding solution, which is now very satisfactory, is graphed every 10 units in t in Fig. 1. The analytical and numerical solutions are almost indistinguishable. The computational statistics as well as the average value of the L_2-norm of the error are summarized in Table 1. With a classical 7-point centered finite difference scheme, accuracy slightly improves at the price of larger computational costs. However, for longer integration times, the use of a 7-point centered scheme leads to failure of the run at about $t = 673$, whereas stagewise differentiation allows the run to be continued up to $t = 1600$, where a slight phase lag between the numerical and analytical solutions appears.

3.2. The "good" Boussinesq equation

The French scientist Joseph Boussinesq (1842–1929) described in the 1870s model equations for the propagation of long waves on the surface of water. The so-called "good" Boussinesq equation

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0 \tag{9}$$

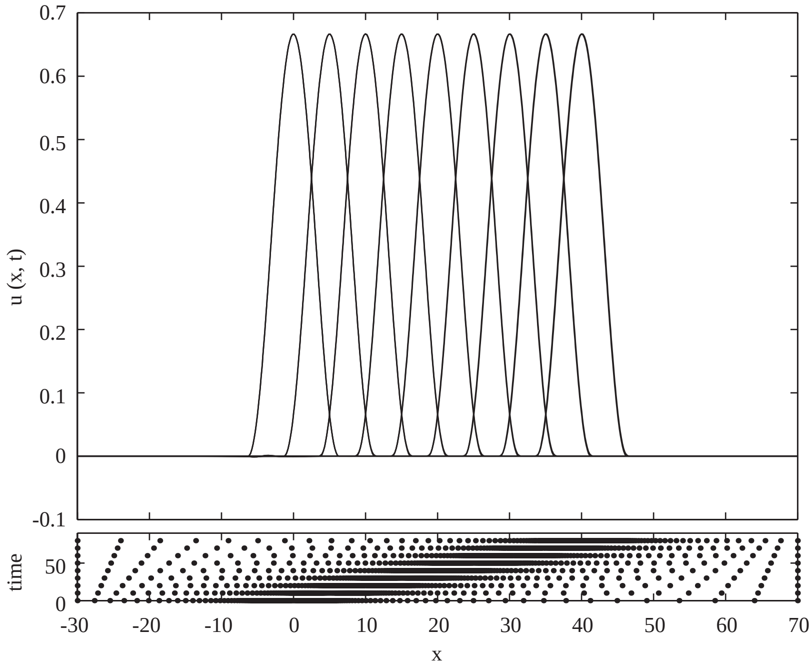

Fig. 1. Propagation of a compacton every 10 units in t—numerical solution on an adaptive grid using stagewise differentiation and exact solution (superimposed).

admits solitary wave solutions of the form

$$u(x,t) = \frac{3(1-c^2)}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{1-c^2}}{2}(x-ct)\right) \tag{10}$$

for wave speed $|c| < 1$.

This problem can be solved relatively easily on a fixed uniform grid with $N = 501$ points, and stagewise differentiation of the fourth-order term (i.e., successive application of a 5-point centered scheme for computing a second-order derivative, $u_{xxxx} = (u_{xx})_{xx}$). To this end, the second-order (in time) PDE (9) is decomposed into two first-order PDEs, through a standard change of variables $v_1 = u,\ v_2 = u_t$. The smooth solution profiles do not require the use of AMR (except, if we had to solve the problem for a long time period, i.e., a large spatial domain). The numerical and analytical solutions for $c = 0.75$ are shown in Fig. 2 on the time interval $(0, 50)$.

3.3. The Kaup–Kupershmidt equation

We consider the fifth-order KK equation

$$u_t + 10uu_{xxx} + 25u_xu_{xx} + 20u^2u_x + u_{xxxxx} = 0, \tag{11}$$

which is one of the solitonic equations related to the integrable cases of the Hénon–Heiles system.

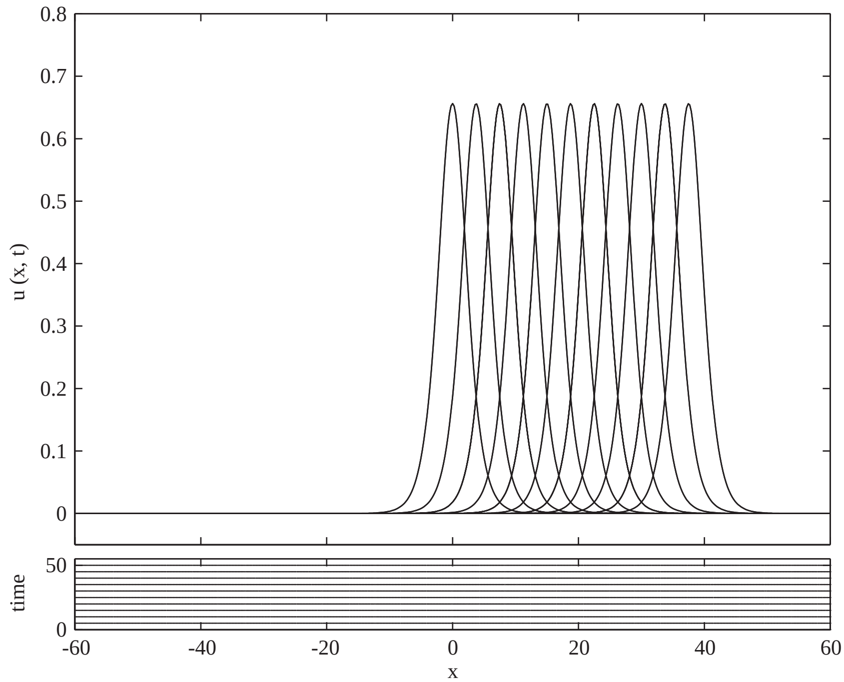

Fig. 2. Boussinesq: propagation of a soliton every 5 units in t—numerical solution on a fixed uniform grid using stagewise differentiation and exact solution (superimposed).

Table 2
Multi-soliton solutions of the KK equation: computational statistics

Soliton	N	Approx.	N_{adapt}	FNS	JACS	STEPS	CPU (s)
1	285–303	$((((u_x)_x)_x)_x)_x$	1	14091	1763	3516	51
2	446–561	$((((u_x)_x)_x)_x)_x$	1	17768	2222	4436	114
3	503–803	$((((u_x)_x)_x)_x)_x$	1	17590	2200	4390	151

Using a simplified version of Hirota's method and symbolic manipulation tools, Hereman and Nuseir [4] derived explicit forms of multi-soliton solutions to the KK equation. Here, we consider the one-soliton solution on the time interval $(0, 50)$ (Fig. 3), the two-soliton solution on the time interval $(-10, 14)$ (Fig. 4(a)) and the three-soliton solution on the time interval $(-5, 5)$ (Fig. 4(b)). These numerical solutions, which are almost indistinguishable from the exact solutions, are obtained at very reasonable computational costs using AMR and stagewise differentiation (the tuning parameters take the following values: $\alpha = 0$, $c = 0.01$, $K = 1.03$ and $N_{adapt} = 1$). The computational statistics are given in Table 2.

On the other hand, these problems are very difficult to solve on a fixed uniform grid. For instance, in the one-soliton case, the use of direct differentiation on a fixed uniform grid with 5001 points leads to an early failure of the run (at $t \approx 0.012$). The use of stagewise differentiation on the same

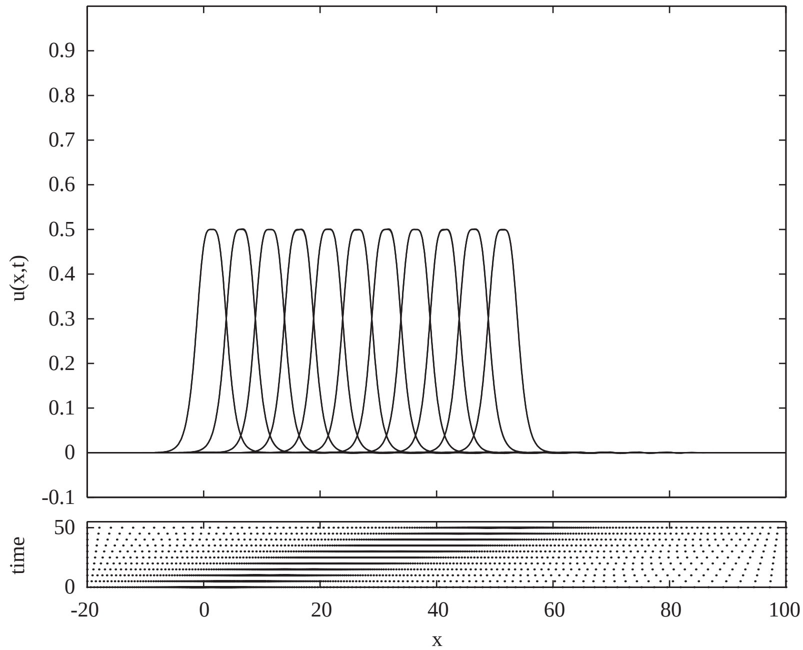

Fig. 3. KK: propagation of a soliton every 5 units in t—numerical solution on an adaptive grid using stagewise differentiation and exact solution (superimposed).

grid allows the run to be performed on a somewhat longer time interval ($t \approx 7.1$). However, many more than 5001 grid points would be required to achieve a successful run on the entire time interval $(0, 50)$. The use of AMR allows this number to be reduced to an average of about 300. On the highly nonuniform grids, the selection of suitable approximations for the high-order derivative terms appears particularly delicate. Again, the use of direct differentiation leads to an early failure of the run (at $t \approx 0.0026$), whereas the use of stagewise differentiation is quite successful (as depicted in Fig. 3).

3.4. An extended KdV5 equation

Champneys and Groves [1] studied the global existence properties of solitary wave solutions to the extended KdV5 equation

$$u_t + \tfrac{2}{15} u_{xxxxx} + (u - b)u_{xxx} + (3u + 2u_{xx})u_x = 0. \tag{12}$$

This equation, which is used in the study of water waves with surface tension, possesses an explicit solitary wave solution in the form

$$u(x,t) = 3\left(b + \tfrac{1}{2}\right) \operatorname{sech}^2\left(\sqrt{\frac{3(2b+1)}{4}}(x + at)\right), \quad a = \frac{3}{5}(2b+1)(b-2),\ b \geqslant -\frac{1}{2}. \tag{13}$$

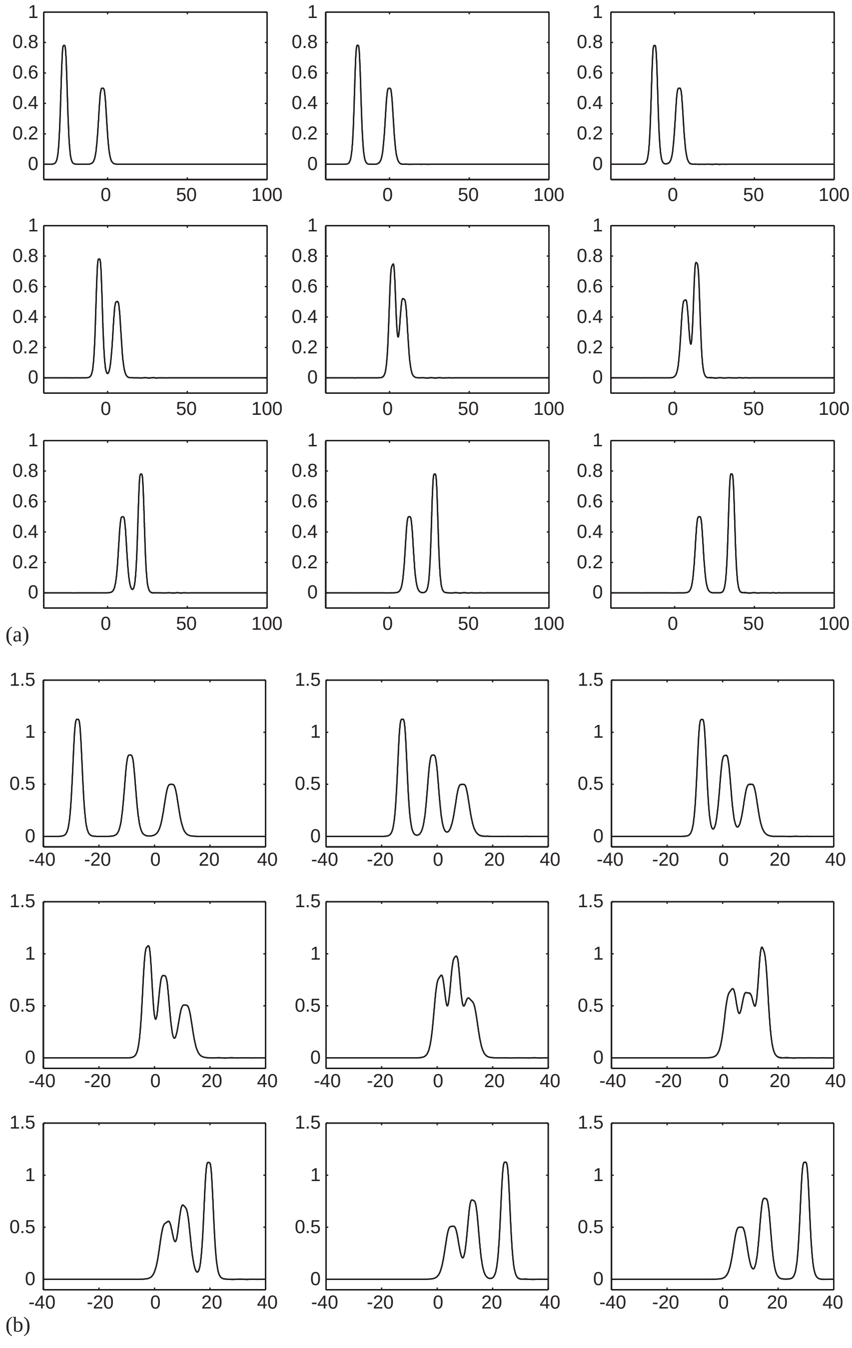

Fig. 4. KK: interaction between two solitons every 3 units in t(a) and interaction between three solitons every 1 unit in t(b)—numerical solution on an adaptive grid using stagewise differentiation and exact solution (superimposed).

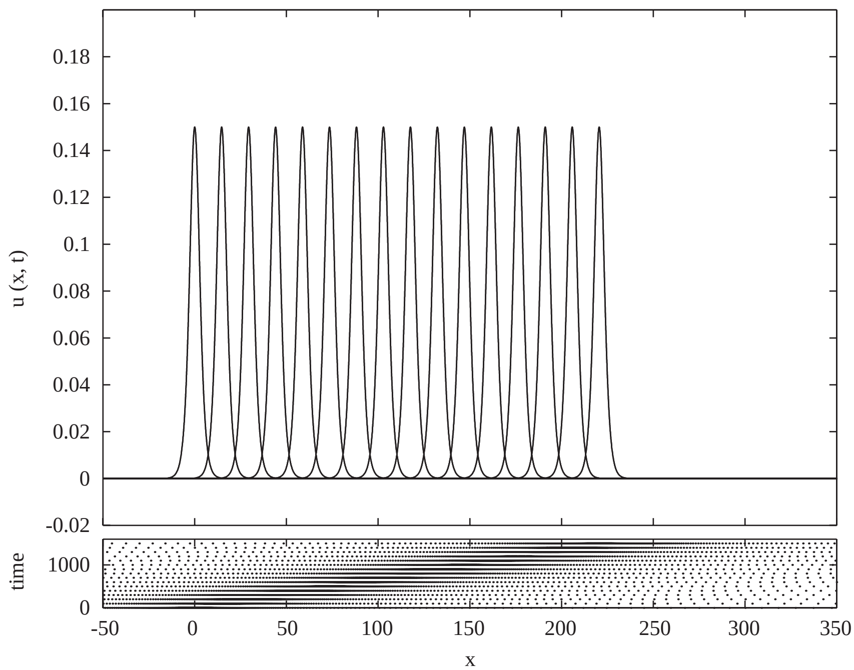

Fig. 5. Extended KdV5: propagation of a soliton every 100 units in t—numerical solution on an adaptive grid using stagewise differentiation and exact solution (superimposed).

We study the propagation of a soliton on the time interval $(0, 1500)$ for the special case $b = -0.45$. The observations are similar to those made for the KK equation:

- direct differentiation on a fixed uniform grid with 5001 points leads to failure of the run at $t \approx 58.8$;
- stagewise differentiation on the same grid allows a longer simulation run to be achieved, i.e. failure occurs at $t \approx 456.6$;
- AMR with direct differentiation leads to failure of the run at $t \approx 0.0002$;
- AMR with stagewise differentiation is successful (see Fig. 5). The tuning parameters take the following values: $\alpha = 0$, $c = 0.0005$, $K = 1.03$ and $N_{\text{adapt}} = 1$). The number of grid points is reduced to about 960.

4. Conclusions

In this study, we consider several solitonic PDE problems. To compute accurate solutions, we use AMR based on the equidistribution principle and spatial regularization.

For the approximation of high-order derivatives on the resulting highly nonuniform spatial grids, we find stagewise differentiation, which computes high-order derivatives by successive computation of lower-order derivatives, to be particularly effective. The use of stagewise schemes appears particularly beneficial for problems involving odd derivatives, e.g. u_{xxx} or u_{xxxxx}, which have large oscillatory modes on the imaginary axis. The approximation of these high-order odd derivatives by stagewise differentiation limits the eigenvalue spectrum as compared to a direct differentiation. In turn, this "limited-bandwidth" approximation probably mitigates the effect of high-frequency numerical noise generated in the course of computation. However, this issue requires a more thorough analysis.

AMR and stagewise differentiation allow numerical solutions to some difficult solitonic PDE problems to be computed, where other conventional techniques perform poorly or even fail.

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