Geometric Aspects of Certain Classes of Nonnegative Matrices

Jan Brandts
Korteweg-de Vries Instituut voor Wiskunde
Universiteit van Amsterdam

(joint work with A. Cihangir, NWO Project 613.001.019)

12 November 2015

Background

- 1985 preview day at UU
- 1986-1990 Mathematics studies UU
- approx. theory, num. lin. alg. (lecturer: Prof. G.L.G. Sleijpen)
- multigrid, finite elements (lecturer: Mr. Sleijpen)
- December 21, 1990: Thesis (supervisor: Gerard Sleijpen)
- 1991-1995 Mathematics PhD UU
- January 16, 1995: PhD defense (thesis co-supervisor: Gerard)

Background

- 1985 preview day at UU (a guy with beard (from the south))
- 1986-1990 Mathematics studies UU
- approx. theory, num. lin. alg. (lecturer: Prof. G.L.G. Sleijpen)
- multigrid, finite elements (lecturer: Mr. Sleijpen)
- December 21, 1990: Thesis (supervisor: Gerard Sleijpen)
- 1991-1995 Mathematics PhD UU
- January 16, 1995: PhD defense (thesis co-supervisor: Gerard)

Background

(lino cuts by Henk van der Vorst)

01. Simplices, Facets, and Dihedral Angles

n-simplex S:

convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$

facet F_v opposite a vertex v of S:

convex hull of the n vertices of S other than v

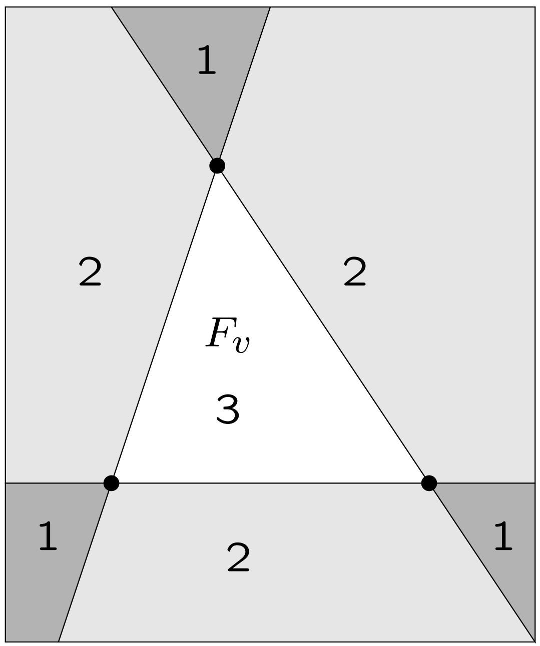
dihedral angle γ between two facets F and G of S:

π minus the angle between outward normals to F and G

An n-simplex has $\binom{n}{2}$ dihedral angles

02. Acute, right, and obtuse dihedral angles

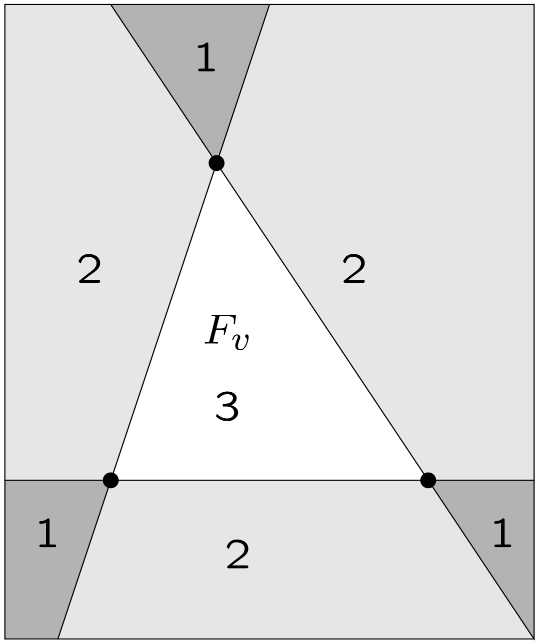
Each facet makes at least one acute angle with another facet



If v projects in the interior of F_v then F_v makes acute angles only

03. Nonobtuse simplices have nonobtuse facets

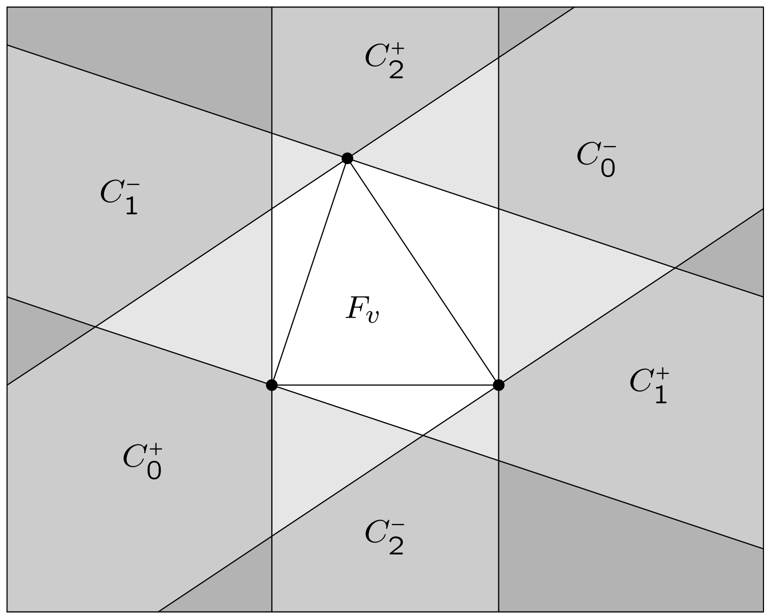
$\boxed{S \text{ is a nonobtuse simplex}}$ if each vertex v projects onto F_v



$\boxed{\text{If } S \text{ is nonobtuse, then } F_v \text{ is also nonobtuse for all vertices } v}$

04. Nonobtuse facets do not yield nonobtuse simplices

if all facets F_v of S are nonobtuse, S may fail to be nonobtuse



S has nonobtuse facets iff each v projects onto the dual hull F_v^*

05. Problem formulation

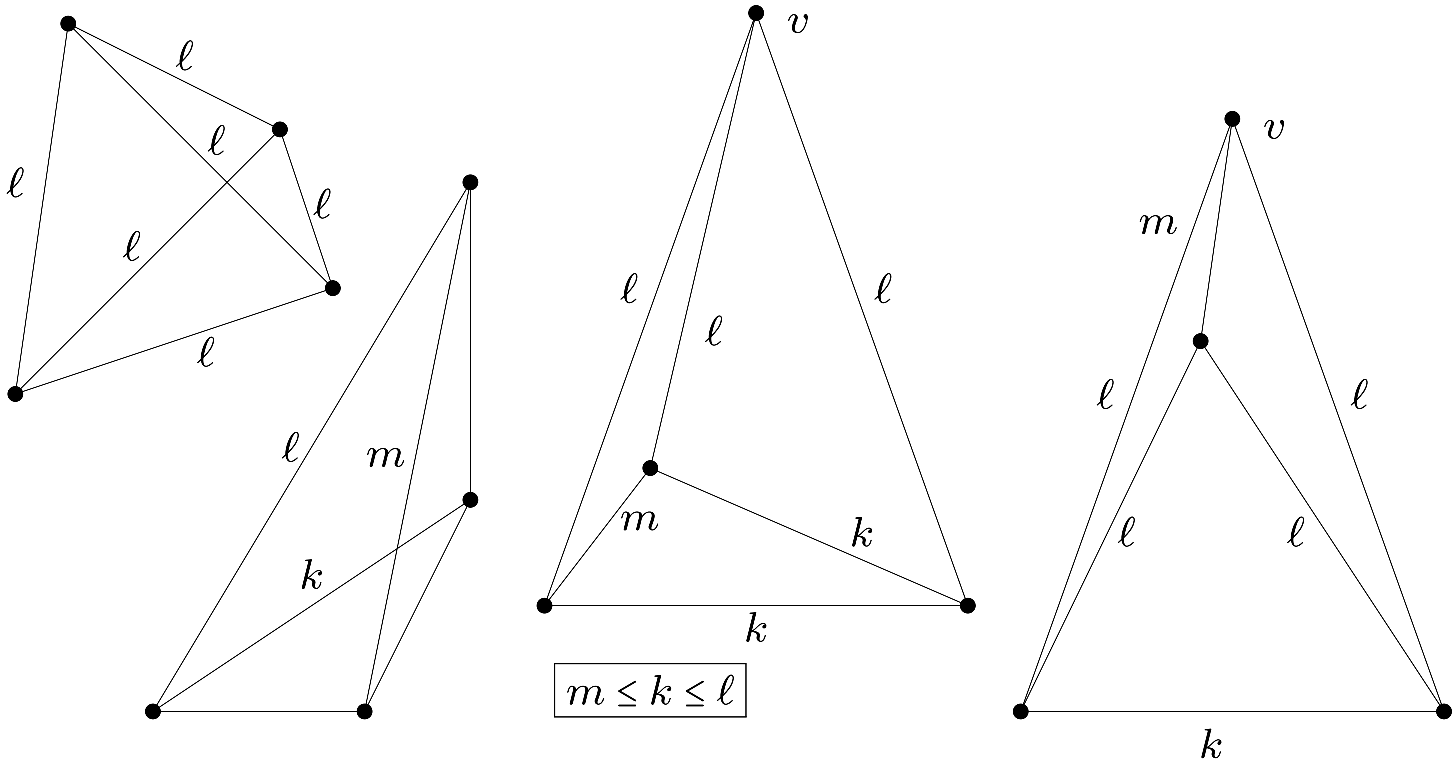
Property $\mathbf{N}$ of S is transferred to all descendants of S

Groal: to find a property $\mathbf{S}$ such that:
• if all descendants of S have property $\mathbf{S}$, then so has S
• $\mathbf{S}$ implies $\mathbf{N}$

Consequence:

If all k-facets of an n-simplex S have property $\mathbf{S}$, then S has property $\mathbf{N}$

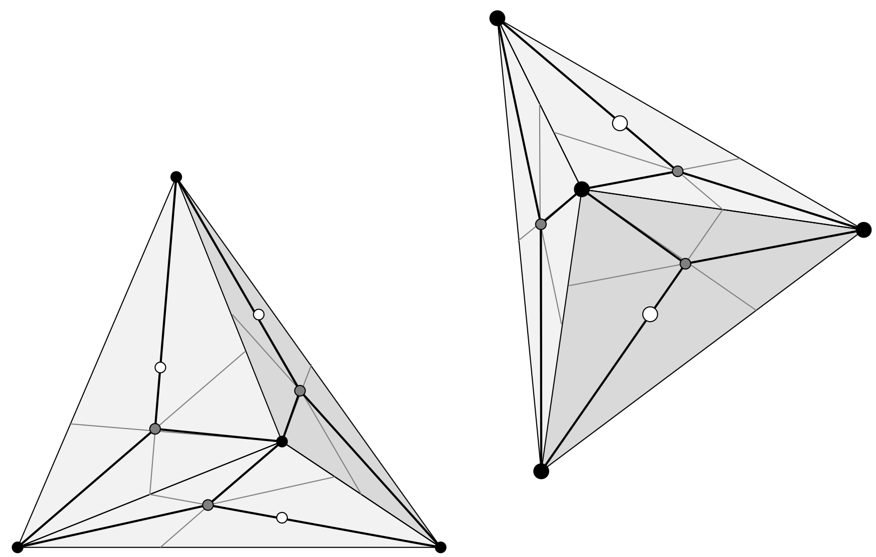
06. Equilateral triangles, Schläfli tets, ultrametric tets



Each of these is an example of a sub-orthocentric tetrahedron

07. Sub-orthocentric tetrahedra

Suppose that each vertex v of a tet $\boxslash$ projects between the orthocenter and a vertex of F_v

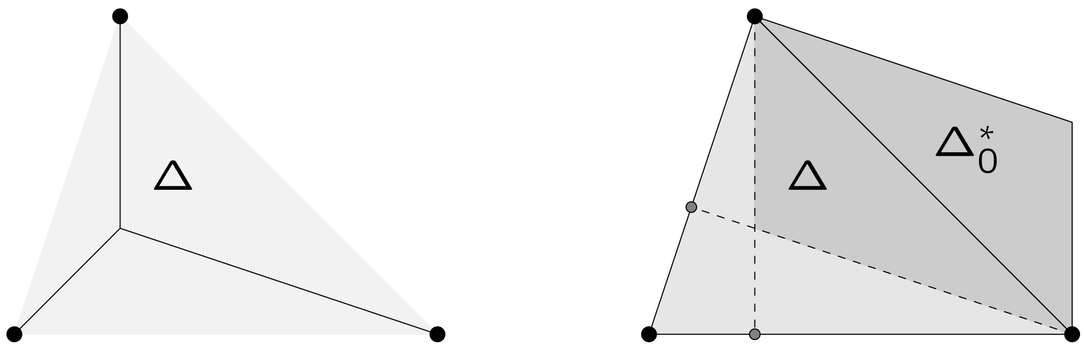


Such a tetrahedron is called sub-orthocentric

A simplex with sub-orthocentric tetrahedral facets is nonobtuse

08. Sub-orthocentric set of a simplex

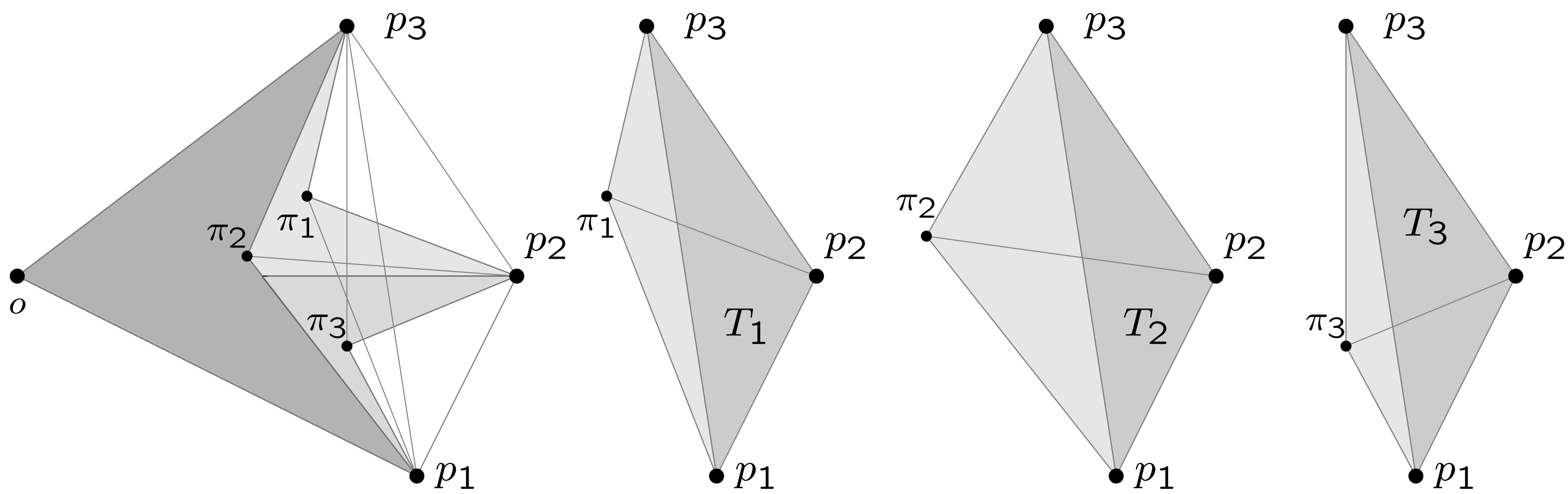
It is not clear how to generalize the sub-orthocentric set to higher dimensions, as n-simplices for $n \geq 3$ do not have orthocenters.



Each altitude of S induces a subdivision of S into n sub-simplices

For each facet F, remove from S the interior of the intersection of all these subsimplices that have F as a facet

09. Example: sub-orthocentric set of a tetrahedron

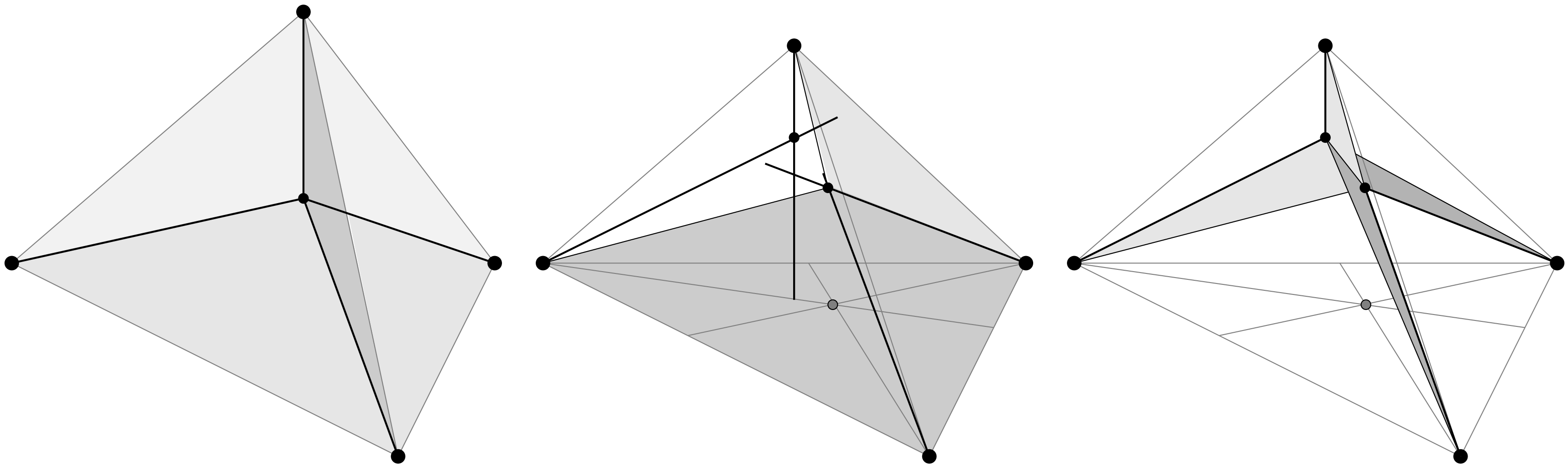


Each altitude of $\boxslash$ induces a subdivision of $\boxslash$ into 3 sub-tetrahedra.

For each triangular facet $\triangle$, remove from $\boxslash$ the iterior of the intersection of all three sub-tetrahedra that have $\triangle$ as a facet

10. Example: sub-orthocentric set of a tetrahedron

Left: the sub-orthocentric set of a orthocentric tetrahedron



Middle and right:

The sub-orthocentric set of a sub-orthocentric tetrahedron

11. The sub-orthocentric 4-simplex

A 4-simplex $\boxtimes$ is $\boxed{\text{sub-orthocentric}}$ if each vertex of $\boxtimes$ projects on the $\boxed{\text{sub-orthocentric set}}$ of its opposite facets.

Theorem: If all facets of $\boxtimes$ are sub-orthocentric, then so is $\boxtimes$

Note: A sub-orthocentric simplex is nonobtuse (by definition).

Conjecture:

$\boxed{\text{If all facets of an } n\text{-simplex } S \text{ are sub-orthocentric, then so is } S}$

12. The linear algebraic context

Let $A \in \mathbb{R}^{n\times n}_{\text{spd}}$ be $\boxed{\text{symmetric positive definite}}$

Then both A and A^{-1} are Gramians

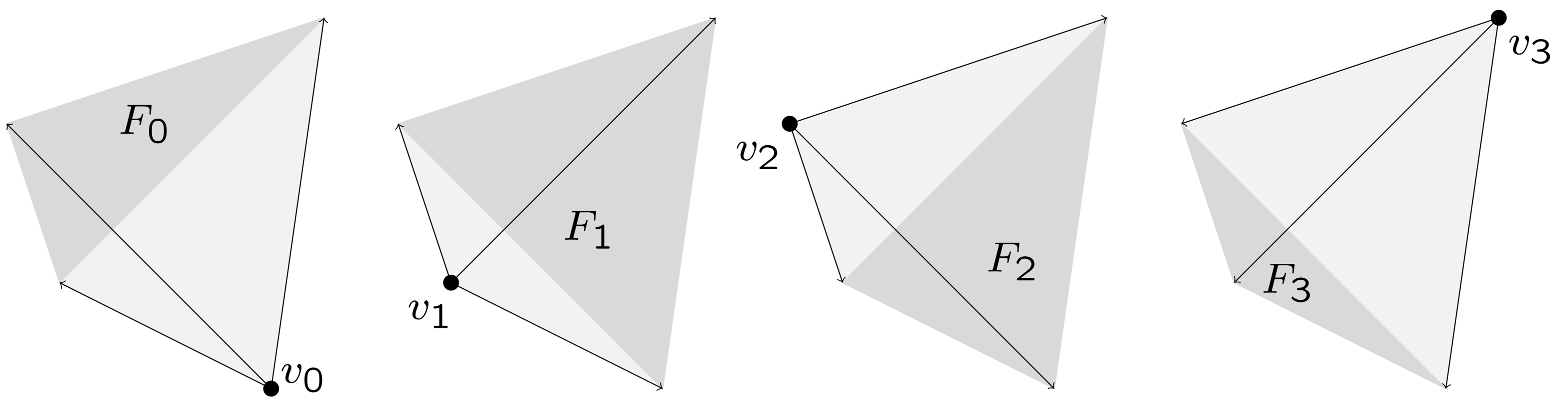
$$\boxed{A = P^\top P} \text{ and } \boxed{A^{-1} = Q^\top Q} \text{ with } \boxed{Q^\top P = I}$$

and with $P \in GL_n(\mathbb{R})$.

If the origin and columns of P are the vertices of an n-simplex S, then the columns of Q and minus their sum, are inward normals to the facets of S

13. Vertex Gramians of a simplex

Let S be an n-simplex with vertices $v_0, \ldots, v_n \in \mathbb{R}^n$.



Definition: A $\boxed{\text{vertex Gramian}}$ G_ℓ of S associated with v_ℓ is the Gramian $G_\ell = P_\ell^\top P_\ell$ of P_ℓ, where P_ℓ has columns $\{v_k - v_\ell \,|\, k \neq \ell\}$

14. Matrix classes related special types of simplices

A simple, but not widely known characterization is:

$$\boxed{\text{The set of all vertex Gramians of nonobtuse simplices}}$$
$$=$$
$$\boxed{\text{The set of all diagonally dominant inverse M-matrices}}$$

($A^{-1} = D - C$ with $C, D \geq 0$ and D diagonal)

Submatrices inherit this property but the converse doet not hold

15. Matrix classes related special types of simplices

Another simple, but not widely known characterization is:

$$\boxed{\text{Simplices with nonobtuse triangular facets}}$$

$$=$$

$$\boxed{\text{Pointwise diagonally dominant doubly nonnegative matrices}}$$

$$(A \in \mathbb{R}^{n\times n}_{\mathsf{spd}}, A \geq 0, a_{jj} \geq a_{ij})$$

How to recognize the inverse of a nonnegative matrix?

16. Matrix classes related special types of simplices

Another simple, but not widely known characterization is:

$$\boxed{\text{Vertex Gramians of simplices with nonobtuse } (n-1)\text{-facets}}$$

$$\subset$$

$$\boxed{\text{Completely positive matrices}}$$

$$(A = P^\top P \text{ with } P \geq 0)$$

Can the columns of a given matrix be mapped simultaneously and isometrically in the nonnegative orthant?

17. Matrix classes related special types of simplices

$\boxed{\text{Simplices with sub-orthocentric tetrahedral facets}}$

=

$\boxed{\text{Ultrametric matrices}}$

all 3×3 principal submatrices are, modulo simultaneous rows and column permutations

$$\begin{bmatrix} d & b & a \\ b & c & a \\ a & a & f \end{bmatrix}, \text{ with } a \leq b < c \leq d \text{ and } \quad a < f. \tag{1}$$

These ultrametric matrices were our research motivation

References

J.H.Brandts and A. Cihangir (2015)
Geometric Aspects of the Symmetric Inverse M-Matrix Problem
Submitted, and to appear also on ArXive.math: (42 pages)

J.H.Brandts and A. Cihangir (2015)
Enumeration and Investigation of Acute 0/1-simplices modulo the Action of the Hyperoctahedral Group
Submitted, and to appear also on ArXive.math: (50 pages)