Past, present and space-time

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Outline

- Past.
- Past $\rightarrow$ present.
- Space-time.

Past I: 1983-1984

Diploma thesis

Stabiliteit en instabiliteit van een Hopscotchmethode

Deze scriptie vormt een onderdeel van mijn afstudeerfase bij Prof. dr. A.v.d. Sluis aan de Rijksuniversiteit Utrecht. Hierbij wil ik hem en Dr. G.L.G Sleijpen bedanken voor de wijze waarop zij mij hebben weten te motiveren en voor de uitstekende begeleiding die ik van hen heb gekregen.

Arnold Reusken,

Utrecht, Augustus 1984

Past II: 1984-1988 (PhD thesis)

Convergence Analysis of Nonlinear Multigrid Methods

Uit : Proefschrift, 27.10.1988

DANKWOORD

Mijn dank gaat uit naar iedereen die mij op enigerlei wijze gesteund heeft bij het maken van dit proefschrift. Enkelen van hen, wier bijdrage voor mij van bijzondere betekenis is geweest wil ik hier noemen.

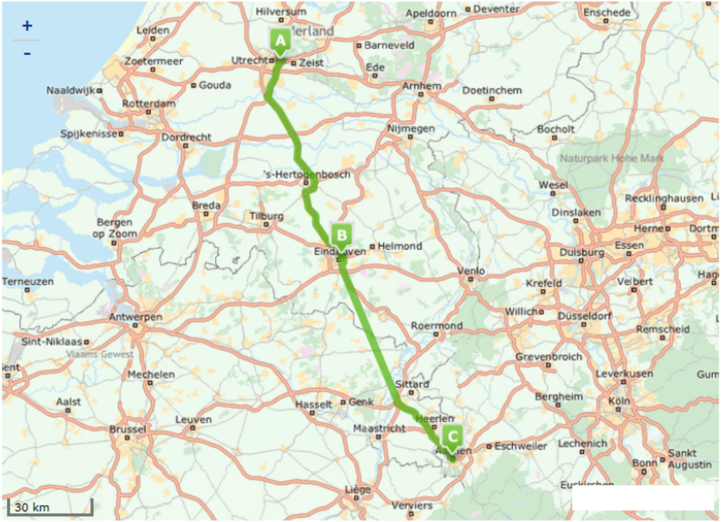
Mijn promotor, Prof.dr. A. van der Sluis, gaf mij de vrijheid om het onderzoek naar eigen inzicht te verrichten en was altijd bereid om mij met raad en daad terzijde te staan. Zijn vragen en kritische opmerkingen waren vaak precies de goede.

Gerard Sleijpen presteerde het menigmaal om zich in mijn problemen te verdiepen en om mijn ideeën met zijn visie te verrijken. !

Van Professor van der Sluis en van Gerard heb ik veel geleerd.

Past → present

A: 1978-1989 UU, B: 1989-1997 TUE C: 1997-present RWTH

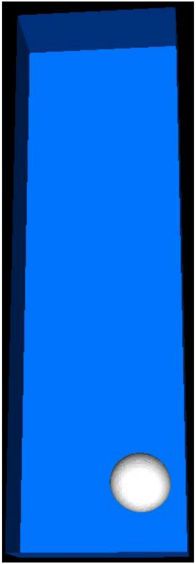


Space-time Finite Element Method for PDEs on Evolving Surfaces

Motivation: simulation of two-phase incompressible flows

system: n-butanol/water

Model: Navier-Stokes equations
$+$ coupling conditions



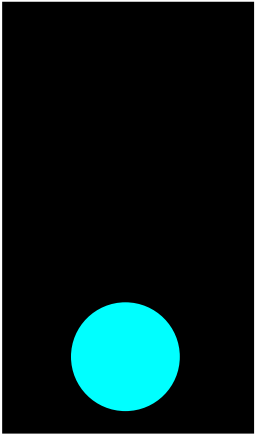
Rising droplet with surfactant transport

- gravity-driven butanol-droplet in water
- Velocity field determined from NS-equations.

$+$ surfactant eqn. $\dot{S} - D_\Gamma \Delta_\Gamma S + (\nabla_\Gamma \cdot \mathbf{u}) S = 0$

Elliptic PDE on evolving surface

solution



Surfactant PDE

$\Gamma(0)$ smooth surface in $\mathbb{R}^3$, $\partial\Gamma(t) = \emptyset$.

$\Gamma(t)$, $t \in [0, T]$, advected by smooth $\mathbf{w} = \mathbf{w}(\mathbf{x}, t) \in \mathbb{R}^3$.

Model for diffusive mass transport on $\Gamma(t)$:

Diffusion equation

$$\dot{u} + (\mathrm{div}_\Gamma \mathbf{w})u - \Delta_\Gamma u = 0 \qquad \text{on} \quad \Gamma(t), \quad t \in (0, T]$$

with $\dot{u} = \frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u$.
Initial condition $u(\mathbf{x}, 0) = u_0(\mathbf{x})$ for $\mathbf{x} \in \Gamma(0)$.

Weak formulations

Space-time manifold

$$\Gamma_* = \bigcup_{t\in(0,T]} \Gamma(t) \times \{t\}, \quad \Gamma_* \subset \mathbb{R}^4$$

Suitable (Sobolev) spaces W (trial), H (test) on Γ_*.

$$a(u,v) = (\nabla_\Gamma u, \nabla_\Gamma v)_{L^2(\Gamma_*)}, \quad u,v \in H.$$

Well-posed weak formulation

determine $u \in W$ such that

$$\langle \dot{u}, v\rangle + a(u,v) = (f,v)_{L^2(\Gamma_*)} \quad \text{for all} \quad v \in H.$$

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Theorem [Olshanskii,R. SINUM 2014]

Weak formulation is well-posed.

Analysis based on continuity and inf-sup property

A time-discontinuous Eulerian weak formulation

Time slabs: $t_j = j\Delta t$, $I_n := (t_{n-1}, t_n]$, $\Gamma_*^n := \cup_{t \in I_n} \Gamma(t)$.

Broken space $W^b := \oplus_{n=1}^N W_n$.

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$$d(u,v) = \sum_{n=1}^N d^n(u,v), \quad d^n(u,v) = \int_{\Gamma(t_{n-1})} [u]^{n-1} v_+^{n-1} \, ds$$

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$$\langle \dot{u}, v \rangle_b = \sum_{n=1}^N \langle \dot{u}_n, v_n \rangle_n$$

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Time-discontinuous weak formulation (allows time-stepping)

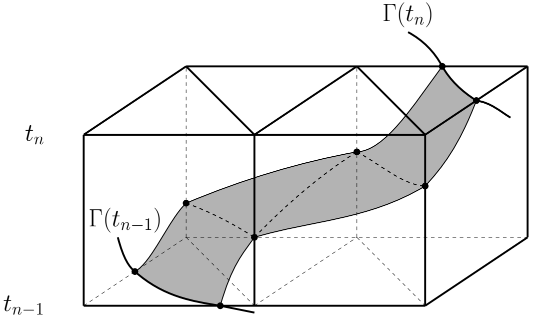
Find $u \in W^b$ such that

$$\langle \dot{u}, v \rangle_b + a(u,v) + d(u,v) = F(v) \quad \text{for all} \quad v \in W^b.$$

Galerkin FEM based on time-discontinuous formulation

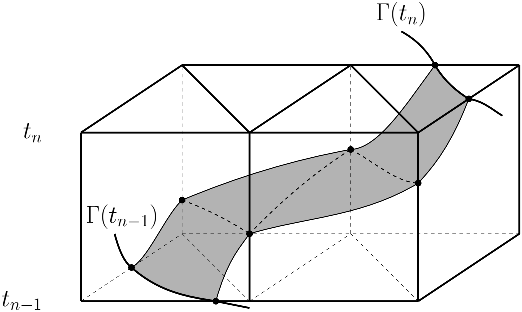
Key ideas:

- W^b replaced by FE space W_h^Γ on Γ_*.
- For FE space we use trace of standard outer space-time FE space.
- Γ_*^n is approximated (zero level of discrete level set function).



Trace FE spaces

Space-time slab: $Q^n = \Omega \times (t_{n-1}, t_n] \subset \mathbb{R}^{d+1}$.
$\mathcal{T}_n$: triangulation of Ω.
V_n: standard FE space on $\mathcal{T}_n$ (piecewise linears).



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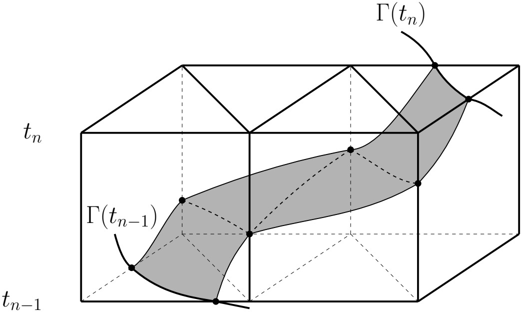

$$W_{n,h} := \{\, w : Q^n \to \mathbb{R} \mid w(x,t) = \phi_0(x) + t\phi_1(x), \quad \phi_0, \phi_1 \in V_n \,\}$$

$$W_{n,h}^{\Gamma} := \{\, v : \Gamma_*^n \to \mathbb{R} \mid v = w_{|\Gamma_*^n}, \quad w \in W_{n,h,\Delta t} \,\}, \quad 1 \leq n \leq N.$$

$$W_h^{\Gamma} := \oplus_{n=1}^{N} W_{n,h}^{\Gamma}$$

Galerkin FEM

Galerkin trace-FEM

Find $u_h = u_{h,\Delta t} \in W_h^\Gamma$ such that

$$\langle \dot{u}_h, v_h \rangle_b + a(u_h, v_h) + d(u_h, v_h) = F(v_h) \quad \text{for all} \quad v_h \in W_h^\Gamma$$

Galerkin FEM

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Time-stepping procedure

For $n = 1, \ldots, N$, do:

Find $u_h = u_{h,\Delta t} \in W_{n,h}^\Gamma$ such that

$$\langle \dot{u}_h, v_h \rangle_n + a^n(u_h, v_h) + \int_{t_{n-1}} u_h^+ v_h^+ \, ds = \int_{t_{n-1}} u_h^{n-1} v_h^+ \, ds + F(v_h) \;\; \forall\, v_h \in W_{n,h}^\Gamma$$

Error analysis [Olshanskii,AR, SINUM 2014]

$$|\!|\!| u |\!|\!|_h^2 := \|u\|_H^2 + \max_{n=1,\dots,N} \|u_-^n\|_{t^n}^2 + \sum_{n=1}^{N} \|[u]^{n-1}\|_{t_{n-1}}^2 .$$

Discrete stability

$$\inf_{u \in W^b} \sup_{v \in W^b} \frac{\langle \dot{u}, v \rangle_b + a(u,v) + d(u,v)}{|\!|\!| v |\!|\!|_h |\!|\!| u |\!|\!|_h} \geq c_s$$

Global stability result. No conditions on Δt.

Error analysis [Olshanskii,AR, SINUM 2014]

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Discrete stability

$$\inf_{u \in W^b} \sup_{v \in W^b} \frac{\langle \dot{u}, v \rangle_b + a(u,v) + d(u,v)}{\|\!|v\|\!|_h \|\!|u\|\!|_h} \geq c_s$$

Global stability result. No conditions on Δt.

Discretization error bounds. Assume $\Delta t \sim h$

$$\|\!|u - u_h\|\!|_h \leq ch\big(\|u\|_{H^2(\Gamma_*)} + \sup_{t \in [0,T]} \|u\|_{H^2(\Gamma(t))}\big).$$

$$\|u - u_h\|_{H^{-1}(\Gamma_*)} \leq ch^2\big(\|u\|_{H^2(\Gamma_*)} + \sup_{t \in [0,T]} \|u\|_{H^2(\Gamma(t))}\big).$$

Experiment: diffusion on a moving+deforming ellipsoid

$\Gamma(t)$ is zero level of

$$\phi(x, y, z, t) = \Big(\frac{x}{1 + 0.25\sin(t)}\Big)^2 + y^2 + z^2 - 1, \quad t \in [0, 4].$$

Velocity field $\mathbf{w}(x, y, z, t) = 0.25\frac{\cos(t)}{1+0.25\sin(t)}\,(x, 0, 0)^T$.

Solution prescribed: $u(x, y, z, t) = e^{-t}xy$.

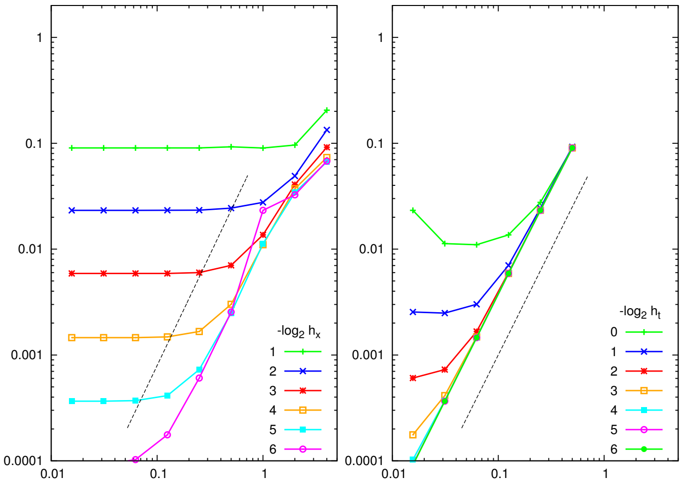
Structure of the algorithm

- Coarse regular tetrahedral triangulation of $\Omega = [-1.5, 1.5]^3$.
  FE space V_h: piecewise linears.

Per time slab

- Local spatial refinement close to Γ_*^n.
- Outer space-time FE space; linear in t.
- Approximation of space-time surface Γ_*^n.
- FE space on Γ_*^n: trace space.
- Apply Galerkin discretization with this FE space.

Results: discrete $L_t^\infty L_x^2$-errors $\sim \max_{t\in[0,4]} \|e_h\|_{L^2(\Gamma(t))}$



$h_x = h$ constant $\qquad\qquad$ $h_t = \Delta t$ constant.

Observations: very stable method; second order convergence.

Evolving surface with topological singularity

Domain $\Omega = (-3, 3) \times (-2, 2)^2$, $t \in [0, 1]$.

Prescribed level set function ϕ,

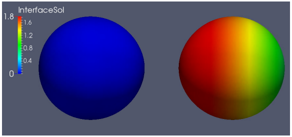
determines $\Gamma(t)$.

Space-time interpolation yields Γ_*^n.

Surfactant transport equation.

$u_0(x) = 3 - x_1$ for $x_1 \geq 0$, zero otherwise.

$h = 1/16, \ \Delta t = 1/128$



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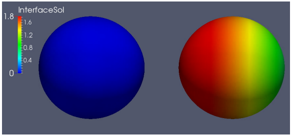

Summary and outlook

- The space-time journey started with the “Hopscotch stability” (1983-1984) under the supervision of Gerard

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Ik wens je alle goeds voor de tijd die komen gaat

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