Twenty years of collaboration...

Gerard's 65th birthday celebration

Martin van Gijzen

Delft University of Technology

November 12, 2015

Twenty years of collaboration...

Outline

Quadratic eigenvalue problems are no problem
Utrecht 1994-1996

Relax to the max
Toulouse 2002-2004

The subspaces are shrinking
Delft 2005-

Quadratic eigenvalue problems are no problem

UTRECHT

Jacobi-Davidson

Jacobi-Davidson

- Jacobi-Davidson was developed by Gerard and Henk when I was working in Utrecht.

Jacobi-Davidson

- Jacobi-Davidson was developed by Gerard and Henk when I was working in Utrecht.
- I met someone at a conference who had difficulties solving a large quadratic eigenvalue problem.

Jacobi-Davidson

- Jacobi-Davidson was developed by Gerard and Henk when I was working in Utrecht.
- I met someone at a conference who had difficulties solving a large quadratic eigenvalue problem.
- At that moment, there was very little research on the numerical solution of nonlinear eigenvalue problems (this has changed completely...).

Jacobi-Davidson

- Jacobi-Davidson was developed by Gerard and Henk when I was working in Utrecht.
- I met someone at a conference who had difficulties solving a large quadratic eigenvalue problem.
- At that moment, there was very little research on the numerical solution of nonlinear eigenvalue problems (this has changed completely...).
- I asked Gerard if it was possible to use JD for this problem.

Jacobi-Davidson

- Jacobi-Davidson was developed by Gerard and Henk when I was working in Utrecht.
- I met someone at a conference who had difficulties solving a large quadratic eigenvalue problem.
- At that moment, there was very little research on the numerical solution of nonlinear eigenvalue problems (this has changed completely...).
- I asked Gerard if it was possible to use JD for this problem.
- The next day he gave me a manuscript of ten pages ...

JD for nonlinear problem $F(\lambda)\mathbf{x} = \mathbf{0}$

a. **Start:**
   Choose initial subspace $\mathbf{V}$

b. **Repeat:**
   (1) Compute desired eigenpair (θ, y) from $\mathbf{V}^* F(\theta)\mathbf{V} y = 0$, $\| y \| =$
   (2) Compute $\mathbf{u} \leftarrow \mathbf{V} y$, $\mathbf{r} \leftarrow F(\theta)\mathbf{u}$
   (3) **Stop if** satisfied
   (4) $\mathbf{w} \leftarrow F'(\theta)\mathbf{u}$
   (5) Solve $(\mathbf{I} - \frac{\mathbf{u}\,\mathbf{w}^*}{\mathbf{w}^*\mathbf{u}})\, F(\theta)\, (\mathbf{I} - \mathbf{u}\,\mathbf{u}^*)\, \mathbf{t} = -\mathbf{r}$
   (6) Expand $\mathbf{V}$: $\mathbf{V} \leftarrow$ Ortho($[\,\mathbf{V}\,|\,\mathbf{t}\,]$).

Application: room acoustics

Propagation of sound:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \triangle p \;\; in \; \Omega$$

c: speed of sound, p: acoustic pressure

Boundary conditions:

- Reflecting wall: $\frac{\partial p}{\partial n} = 0$
- Open wall (pressure release): $p = 0$
- Absorbing wall: $\frac{\partial p}{\partial n} = -\frac{1}{z}\frac{\partial p}{\partial t}$

z is the (complex) impedance, may depend on frequency

Analytical eigenproblem

Assume solution

$$p = \bar{p}e^{\lambda t}$$

with λ an analytical eigenvalue. Substitute in wave equation:

$$\frac{\lambda^2}{c^2}\bar{p} = \triangle \bar{p}$$

B.c. for absorbing wall:

$$\frac{\partial \bar{p}}{\partial n} = -\frac{\lambda}{z(\lambda)}\bar{p}$$

Eigenfrequencies: $f = Im(\lambda)/2\pi$

Discretization with FEM

Discretisation with the Finite Element method gives:

$$\mathbf{K}\mathbf{p} + \lambda \mathbf{C}(\lambda)\mathbf{p} + \lambda^2 \mathbf{M}\mathbf{p} = \mathbf{0}$$

- $\mathbf{p}$: Discretization of $\bar{p}$
- λ: Algebraic eigenvalue
- $\mathbf{K}$: Stiffness matrix, discretization of $-\triangle$
- $\mathbf{C}$: Damping matrix, discretization of impedance condition
- $\mathbf{M}$: Mass matrix, discretization of $\frac{1}{c^2}$

Example cube

- Domain: Ω is $4 \times 4 \times 4 \; m^3$
- 5 reflecting walls
- 1 absorbing wall
- Soundspeed $c = 340m/s$
- Impedance: $z = (0.2 - 1.5i)c$
  (does not depend on frequency $\rightarrow$ quadratic eigenproblem)
- Discretisation: linear tetrahedral elements
- Known analytical eigenvalue: $\lambda = -5.19 + 217.5i$

Results cube

- Grid: $64 \times 64 \times 64 \times 5$ elements $\rightarrow$ 274,625 grid points
- Goal: eigenvalue closest to $-5.19 + 217.5i$
- Selection of Ritz value closest to analytical eigenvalue
- Correction equation 'solved' with 30 GMRES steps
- Restart after 20 JD-iterations to save memory

Calculations performed on Cray T3D using 64 processors.
Eigenvalue $-5.20 + 217.5i$ converged after 33 JD-iterations.

Computing time: 93.4 seconds (20 years ago!!!)

Relax to the max

TOULOUSE

Relaxation

Relaxation strategies

Krylov methods allow the matrix-vector products to be computed to lower accuracy when the process starts to converge.
This was a hot topic at Cerfacs and Jasper van den Eshof and Gerard were also working on it in Utrecht.

Relaxation strategies

Krylov methods allow the matrix-vector products to be computed to lower accuracy when the process starts to converge.
This was a hot topic at Cerfacs and Jasper van den Eshof and Gerard were also working on it in Utrecht.

We assume that an operation is available to compute the matvec in step j to precision η_j:

$$A_{\eta_j}(\mathbf{v}) = \mathbf{A}\mathbf{v} + \mathbf{g} \quad \text{with} \quad \|\mathbf{g}\| < \eta_j \|\mathbf{A}\| \|\mathbf{v}\|$$

Relaxation strategies

Krylov methods allow the matrix-vector products to be computed to lower accuracy when the process starts to converge.
This was a hot topic at Cerfacs and Jasper van den Eshof and Gerard were also working on it in Utrecht.

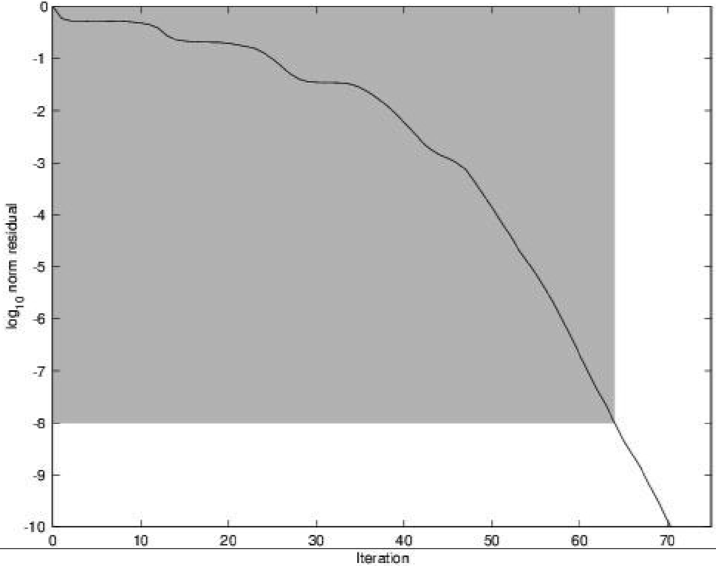
We assume that an operation is available to compute the matvec in step j to precision η_j:

$$A_{\eta_j}(\mathbf{v}) = \mathbf{A}\mathbf{v} + \mathbf{g} \quad \text{with} \quad \|\mathbf{g}\| < \eta_j \|\mathbf{A}\| \|\mathbf{v}\|$$

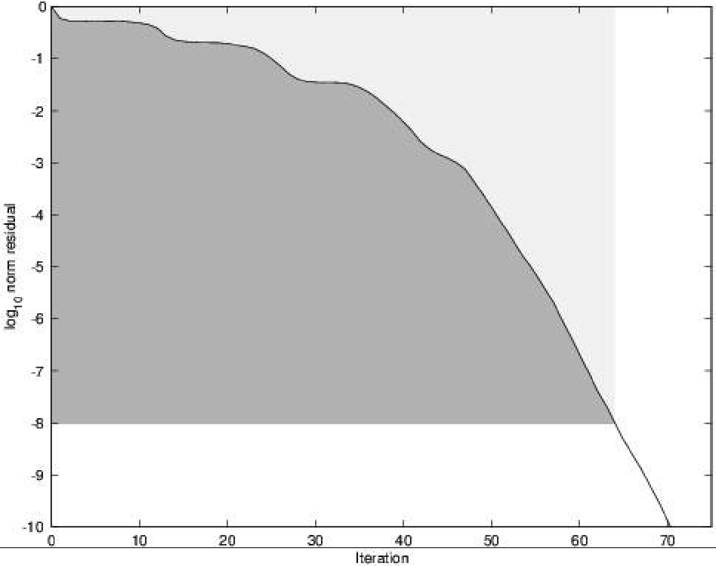
A simple relaxation strategy is to choose

$$\eta_j = \frac{\epsilon}{\|\mathbf{r}_j\|}$$

No relaxation



Relaxation



$\log_{10}$ norm residual
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
0
10
20
30
40
50
60
70
Iteration

Idea: use nested Krylov method

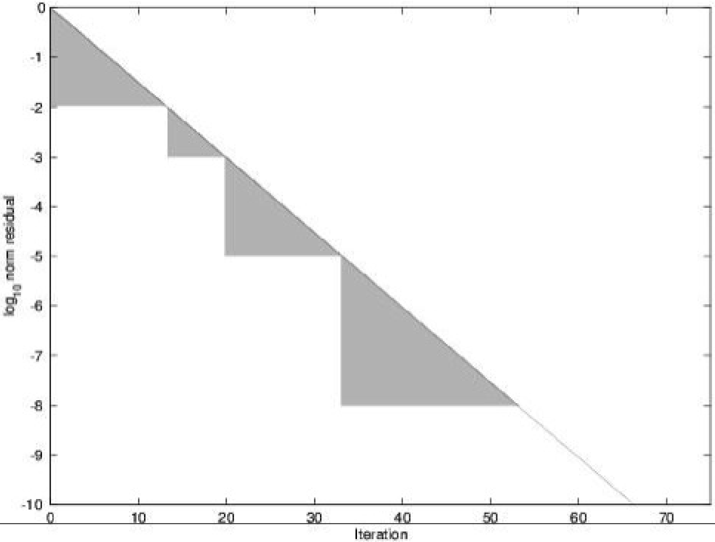
- Outer loop
  - Aim for desired accuracy
  - Use relaxation
- Inner loop (preconditioner)
  - Aim for low accuracy
  - Use relaxation

Idea: use nested Krylov method

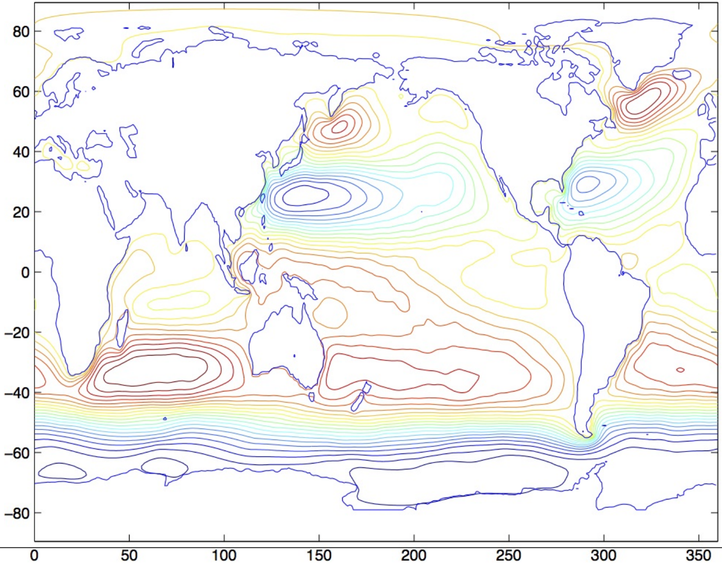
- Outer loop
  - Aim for desired accuracy
  - Use relaxation
- Inner loop (preconditioner)
  - Aim for low accuracy
  - Use relaxation

Almost all matrix-vector products can be performed at low accuracy!

Relax to the max



Example: ocean circulation



Discrete ocean problem

Global system:

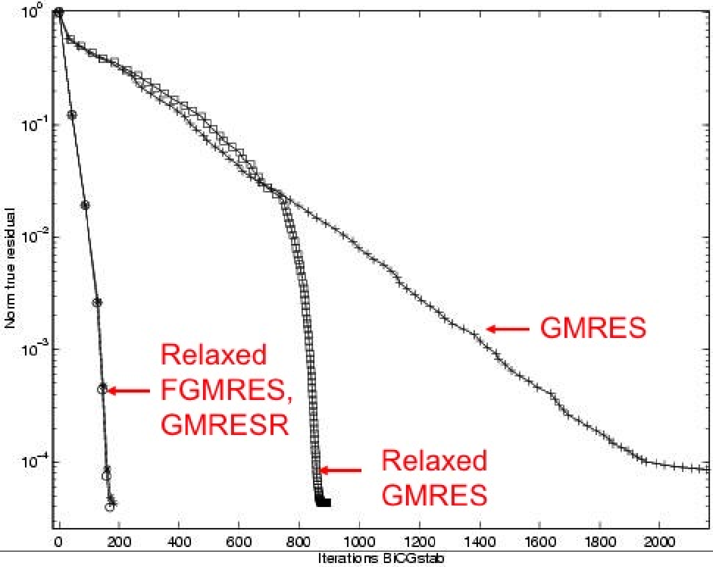
$$\begin{pmatrix} r\mathbf{L} - \mathbf{C} & A\bar{\mathbf{L}} \\ -\bar{\mathbf{L}}^H & \mathbf{M} \end{pmatrix} \begin{pmatrix} \bar{\psi} \\ \bar{\zeta} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}$$

Schur complement for the vorticity:

$$(\mathbf{M} + A\bar{\mathbf{L}}^H (r\mathbf{L} - \mathbf{C})^{-1} \bar{\mathbf{L}})\bar{\zeta} = \bar{\mathbf{L}}^H (r\mathbf{L} - \mathbf{C})^{-1} \mathbf{f}$$

- Action of $(r\mathbf{L} - \mathbf{C})^{-1}$ is approximated by preconditioned Bi-CGSTAB.
- Accuracy of Bi-CGSTAB, and therefore of the multiplication with the Schur complement, can be controlled.

Relax to the max



The subspaces are shrinking

DELFT

The subspaces are shrinking (using IDR)

Two theorems

Theorem (IDR)

Let $\mathbf{P} = [\mathbf{p}_1, \ldots, \mathbf{p}_s]$ *be an* $n \times s$ *matrix and let* (μ_j) *be a sequence in* $\mathbb{C}$. *With* $\mathcal{G}_0 \equiv \mathcal{C}^n$, *define,*

$$\mathcal{G}_{j+1} \equiv (\mu_j \mathbf{I} - \mathbf{A})(\mathcal{G}_j \cap \mathbf{P}^\perp) \qquad (j = 0, 1, \ldots).$$

If $\mathbf{P}^\perp$ *does not contain an eigenvector of* $\mathbf{A}$, *then, for all* $j = 0, 1, \ldots$, *we have that*
1) $\mathcal{G}_{j+1} \subset \mathcal{G}_j$, *and* 2) $\dim \mathcal{G}_{j+1} < \dim \mathcal{G}_j$ *unless* $\mathcal{G}_j = \mathbf{0}$.

Two theorems

Theorem (IDR)

Let $\mathbf{P} = [\mathbf{p}_1, \ldots, \mathbf{p}_s]$ *be an* $n \times s$ *matrix and let* (μ_j) *be a sequence in* $\mathbb{C}$*. With* $\mathcal{G}_0 \equiv \mathcal{C}^n$*, define,*

$$\mathcal{G}_{j+1} \equiv (\mu_j \mathbf{I} - \mathbf{A})(\mathcal{G}_j \cap \mathbf{P}^\perp) \qquad (j = 0, 1, \ldots).$$

If $\mathbf{P}^\perp$ *does not contain an eigenvector of* $\mathbf{A}$*, then, for all* $j = 0, 1, \ldots$*, we have that*
$1)$ $\mathcal{G}_{j+1} \subset \mathcal{G}_j$*, and* $2)$ $\dim \mathcal{G}_{j+1} < \dim \mathcal{G}_j$ *unless* $\mathcal{G}_j = \mathbf{0}$*.*

Theorem (BLOCK-KRYLOV)

Let $\mathbf{P}$*,* (μ_j) *and* $\mathcal{G}_j$ *be as above. Consider the polynomial* p *of degree* j *given by* $p(\lambda) \equiv \prod_{i=1}^{j} (\mu_i - \lambda)$ $(\lambda \in \mathbb{C})$*. Then*

$$\mathcal{G}_j = \{p_j(\mathbf{A})\mathbf{v} \mid \mathbf{v} \perp \mathcal{K}_j(\mathbf{A}^H, \mathbf{P})\}.$$

IDRStab

Can other polynomials than products of linear factors

$$p(\lambda) \equiv \prod_{i=1}^{j} (\mu_i - \lambda)$$

be used (cf. BiCGSTAB versus BiCGstab(ℓ))?

IDRStab

Can other polynomials than products of linear factors

$$p(\lambda) \equiv \prod_{i=1}^{j} (\mu_i - \lambda)$$

be used (cf. BiCGSTAB versus BiCGstab(ℓ))?
Yes, by combining several dimension reductions steps it is possible to construct higher order factors.

The name of this method is **IDRStab**. (SISC 2010)

This is the moment Gerard figured it out

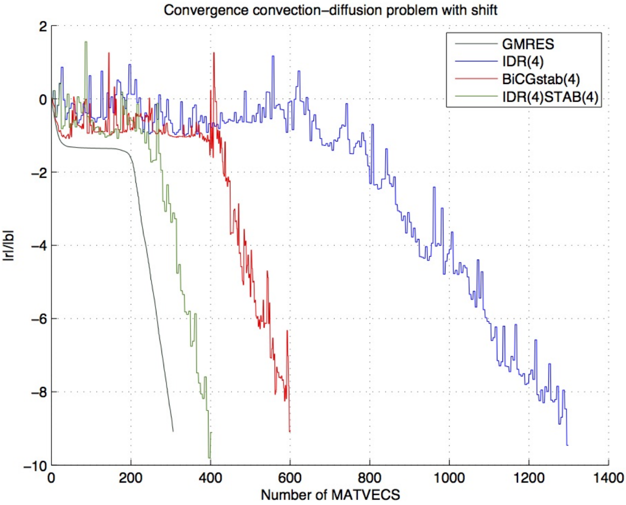
Example

Convection-diffusion problem with shift on unit square:

$$-u_{xx} - u_{yy} + 1000(xu_x + yu_y) + 10u = f \ .$$

- Dirichlet conditions
- Discretised with the finite volume method on a 65 $\times$ 65 grid
- The right-hand side vector such that the solution is one.

Convergence of IDRStab



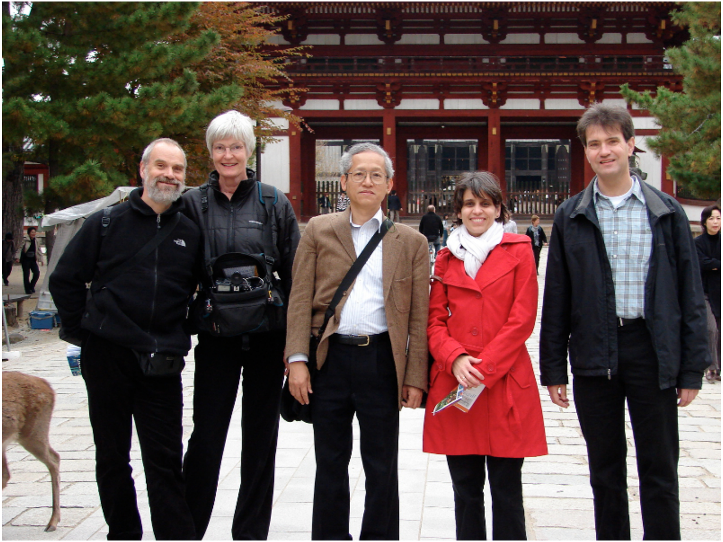
Numerical Linear Algebra course

- For several years we have taught together the Mastermath course on Numerical Linear Algebra

Gerard can teach Numerical Linear Algebra to any audience

I do my best ...

To conclude: nice memories of Japan



Gerard can find the way in any language

Twenty years...

Twenty years...

- of collaboration

Twenty years...

- of collaboration
- but also of friendship and fun

Twenty years...

- of collaboration
- but also of friendship and fun

Gerard: CONGRATULATIONS!