The Laplace Transform

In this section we present the definition and basic properties of the Laplace transform. As a warm-up for the applications with partial differential equations, we will use it to solve some simple ordinary differential equations.

Suppose that $f(t)$ is defined for all $t \geq 0$. The **Laplace transform** of f is the function

As a convention, functions $f, g, \ldots$ are defined for $t \geq 0$ and their transforms $F, G, \ldots$ are defined on the s-axis.

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt. \tag{1}$$

Another commonly used notation for $\mathcal{L}(f)(s)$ is $F(s)$. For the integral to exist f cannot grow faster than an exponential. This motivates the following definition. We say that f is of **exponential order** if there exist positive numbers a and M such that

$$|f(t)| \leq Me^{at} \quad \text{for all } t \geq 0. \tag{2}$$

For example, the functions 1, $4\cos 2t$, $5t\sin 2t$, e^{3t} are all of exponential order. We can now give a sufficient condition for the existence of the Laplace transform.

THEOREM 1 EXISTENCE OF THE LAPLACE TRANSFORM

Suppose that f is piecewise continuous on the interval $[0, \infty)$ and of exponential order with $|f(t)| \leq Me^{at}$ for all $t \geq 0$. Then $\mathcal{L}(f)(s)$ exists for all $s > a$.

Proof We have to show that for $s > a$

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt < \infty.$$

With M and a as above, we have

$$\begin{aligned} \left|\int_0^\infty f(t)e^{-st}\,dt\right| &\leq \int_0^\infty |f(t)|e^{-st}\,dt \leq M\int_0^\infty e^{at}e^{-st}\,dt \\ &= M\int_0^\infty e^{-(s-a)t}\,dt = \frac{M}{s-a} < \infty. \end{aligned}$$

■

Note that the function $\frac{1}{\sqrt{t}}$ is not of exponential order, because of its behavior at $t = 0$. However, we will show in Example 2 below that its Laplace transform $\mathcal{L}\left(\frac{1}{\sqrt{t}}\right)(s)$ exists for all $s > 0$. Thus Theorem 1 provides sufficient but not necessary conditions for the existence of the Laplace transform.

(c) Using (a), and (15), Section 4.7,

$$\mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = \mathcal{L}\left(t^{-1/2}\right) = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}.$$

■

Operational Properties

We will derive in the rest of this section properties of the Laplace transform that will assist us in solving differential equations. We are particularly interested in those formulas involving a function, its transform, and the transform of its derivatives. These formulas are similar to the operational properties of the Fourier transform. Because the Laplace transform is defined by an integral over the interval $[0, \infty)$, some of the formulas will involve the values of the function and its derivatives at 0.

THEOREM 2 LINEARITY

If f and g are functions and α and β are numbers, then

$$\mathcal{L}(\alpha f + \beta g) = \alpha\mathcal{L}(f) + \beta\mathcal{L}(g).$$

The proof is left as an exercise. You should also think about the domain of definition of $\mathcal{L}(\alpha f + \beta g)$ in terms of the domains of definition of $\mathcal{L}(f)$ and $\mathcal{L}(g)$.

EXAMPLE 3 $\mathcal{L}(\cos kt)$ **and** $\mathcal{L}(\sin kt)$

These transforms can be evaluated directly by using (1). Our derivation will be based on Euler's identity $e^{ikt} = \cos kt + i\sin kt$ and the linearity of the Laplace transform. We have

$$\begin{aligned}
\mathcal{L}(\cos kt) + i\mathcal{L}(\sin kt) &= \int_0^\infty (\cos kt + i\sin kt)e^{-st}\,dt \\
&= \int_0^\infty e^{-t(s-ik)}\,dt = -\frac{e^{-t(s-ik)}}{s-ik}\bigg|_0^\infty = \frac{1}{s-ik} \\
&= \frac{s+ik}{s^2+k^2} = \frac{s}{s^2+k^2} + i\frac{k}{s^2+k^2}.
\end{aligned}$$

Equating real and imaginary parts, we get

$$\mathcal{L}(\cos kt) = \frac{s}{s^2+k^2} \quad \text{and} \quad \mathcal{L}(\sin kt) = \frac{k}{s^2+k^2}.$$

For an alternative derivation, see Example 10 below. ■

The next result is very useful. It states that the Laplace transform takes derivatives into powers of s.

THEOREM 3 LAPLACE TRANSFORMS OF DERIVATIVES

(i) Suppose that f is continuous on $[0, \infty)$ and of exponential order as in (2). Suppose further that f' is piecewise continuous on $[0, \infty)$ and of exponential order. Then

(3) $$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)\,.$$

(ii) More generally, if $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and of exponential order as in (2), and $f^{(n)}$ is piecewise continuous on $[0, \infty)$ and of exponential order, then

(4) $$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)\,.$$

Proof Since f is of exponential order, then (2) holds for some positive constants a and M. The transform $\mathcal{L}(f')(s)$ is to be computed for $s > a$. Before we start the computation, note that for $s > a$

$$\lim_{t\to\infty} |f(t)|\,e^{-st} = \lim_{t\to\infty} \underbrace{|f(t)|}_{\le Me^{at}} e^{-at}e^{-(s-a)t} \le M \lim_{t\to\infty} e^{-(s-a)t} = 0,$$

because $s - a > 0$. We now compute, using (1) and integrating by parts,

$$\begin{aligned}
\mathcal{L}(f')(s) &= \int_0^\infty f'(t)e^{-st}\,dt \quad (s > a)\\
&= f(t)e^{-st}\Big|_0^\infty - (-s)\underbrace{\int_0^\infty f(t)e^{-st}\,dt}_{\mathcal{L}(f)(s)}\\
&= -f(0) + s\,\mathcal{L}(f),
\end{aligned}$$

which proves (i). Part (ii) follows by repeated applications of (i). ■

When $n = 2$, (4) gives

(5) $$\boxed{\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)\,.}$$

The following is a counterpart of Theorem 3 showing that the Laplace transform takes powers of t into derivatives.

THEOREM 4 DERIVATIVES OF TRANSFORMS

(i) Suppose $f(t)$ is piecewise continuous and of exponential order. Then

(6) $$\mathcal{L}(t f(t))(s) = -\frac{d}{ds}\,\mathcal{L}(f)(s)\,.$$

(ii) In general, if $f(t)$ is piecewise continuous and of exponential order, then

(7) $$\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}\,\mathcal{L}(f)(s)\,.$$

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	e^{at}	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n e^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$	$\frac{e^{-cs}}{s}$	26.	$\delta(t-c)$	e^{-cs}
27.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$	28.	$u_c(t)g(t)$	$e^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$e^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

the Gamma function:

$$\Gamma(t)=\int_0^\infty e^{-x}x^{t-1}\,dx$$

If n is a positive integer then,

$$\Gamma(n+1)=n!$$

$$\Gamma(p+1)=p\Gamma(p)$$

$$p(p+1)(p+2)\cdots(p+n-1)=\frac{\Gamma(p+n)}{\Gamma(p)}$$

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$