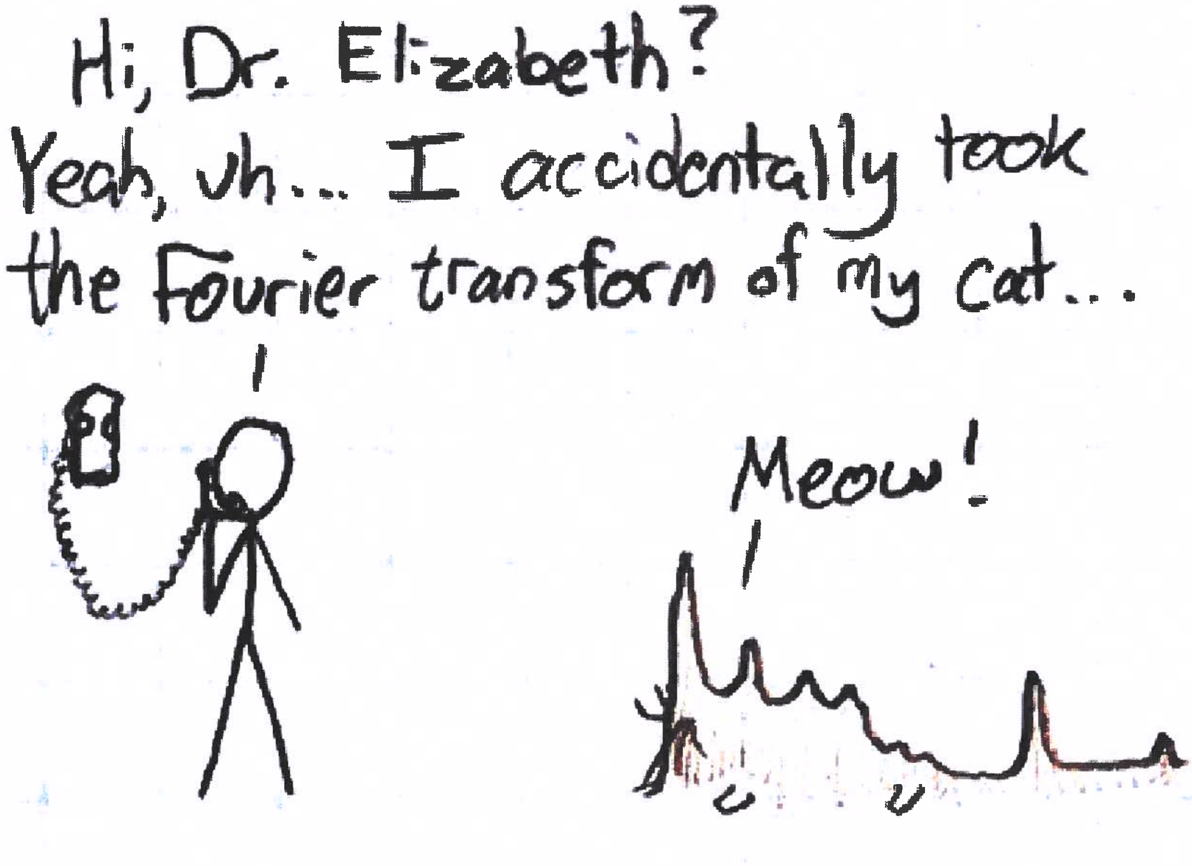
Fouriertransformatie & FT-opgaven



7.1 The Fourier Integral Representation

To help us understand and appreciate the topics of this section, let us recall from Section 2.3 the Fourier series representation theorem. Given a $2p$-periodic function f, we have

$$(1) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right),$$

where

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(t)\, dt,$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(t) \cos \frac{n\pi}{p} t\, dt, \qquad b_n = \frac{1}{p} \int_{-p}^{p} f(t) \sin \frac{n\pi}{p} t\, dt.$$

Now suppose that f is defined on the entire real line but is not periodic. Can we represent f by a Fourier series? It turns out that we no longer have a Fourier series representation, but a Fourier integral representation. The answer is given by the following important theorem.

THEOREM 1 FOURIER INTEGRAL REPRESENTATION

Suppose that f is piecewise smooth on every finite interval and that $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$. Then f has the following **Fourier integral representation**

$$(2) \qquad f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x]\, d\omega \quad (-\infty < x < \infty),$$

where, for all $\omega \geq 0$,

$$(3) \qquad A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t\, dt; \qquad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t\, dt.$$

The integral in (2) converges to $f(x)$ if f is continuous at x and to $\frac{f(x+)+f(x-)}{2}$ otherwise.

Refer to Section 2.2 for the definition of *piecewise smooth*.

Note the similarity between the Fourier integral (2) and the Fourier series (1). The sum in (1) is replaced by an integral in (2), and the integrals from $-p$ to p that define the Fourier coefficients are replaced by integrals from $-\infty$ to ∞ in (3). Also, in (3), the "Fourier coefficients" are computed over a continuous range $\omega \geq 0$, whereas the Fourier coefficients of a periodic function are computed over a discrete range of values $n = 0, 1, 2, \ldots$.

As with Fourier series, for Theorem 1 to hold we imposed sufficient conditions on f, including the condition

$$\int_{-\infty}^{\infty} |\, f(x)\, |\, dx < \infty,$$

which is expressed by saying that f is **integrable on the entire real line.** This condition also ensures the existence of the improper integrals defining $A(\omega)$ and $B(\omega)$ in (3).

We will omit the proof of Theorem 1, which involves ideas similar to those in the proof of the Fourier series representation theorem (see [1]). The fact that the series in (1) is changed into an integral in the Fourier series representation as the period tends to infinity can be motivated as follows.

Suppose that f is an integrable function on the real line. Restrict f to a finite interval $(-p, p)$. Take the part of f that is inside $(-p, p)$ and extend it periodically outside this interval. The periodic extension agrees with f on $(-p, p)$ and has a Fourier series as in (1), which represents $f(x)$ for x in $(-p, p)$. The question now is, What happens to this representation as $p \to \infty$? To answer this question, let us investigate the Fourier coefficients as $p \to \infty$. Since f is integrable, it follows that $a_0 \to 0$ as $p \to \infty$. Also, we can draw a connection between a_n and b_n and $A(\omega)$ and $B(\omega)$ as follows. The integrability of f implies that the integrals in (3) can be approximated by merely integrating over the (large) finite interval $(-p, p)$. The difference is just the tail ends of the integrals, which can be made arbitrarily small. Thus, for large p, we can write

$$a_n \approx \frac{1}{p} \int_{-\infty}^{\infty} f(t) \cos \frac{n\pi}{p} t \, dt = A(\omega_n)\, \Delta\omega \qquad \text{(by (3))},$$

where $\omega_n = (n\pi)/p$ and $\Delta\omega = \frac{\pi}{p}$. Similarly, $b_n \approx B(\omega_n)\, \Delta\omega$. Plugging these values into (1), we see that for very large p, we have

$$f(x) \approx \sum_{n=1}^{\infty} \left(A(\omega_n) \cos \omega_n x + B(\omega_n) \sin \omega_n x\right) \Delta\omega. \tag{4}$$

We have conveniently used a notation that suggests that the sum in (4) is a Riemann sum. This sum samples the integrand of (2) at equally spaced points ω_n with a partition size $\Delta\omega$ being precisely the distance between two consecutive ω_n. The fact that n goes to infinity in (4) indicates that this Riemann sum is not over a finite interval but (regardless of p) spans the entire nonnegative ω-axis. As $p \to \infty$, $\Delta\omega \to 0$ and this Riemann sum converges to the integral in ω, given by (2).

EXAMPLE 1 A Fourier integral representation

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution From (3),

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt = \frac{1}{\pi} \int_{-1}^{1} \cos \omega t \, dt = \left[\frac{\sin \omega t}{\pi\omega}\right]_{-1}^{1} = \frac{2 \sin \omega}{\pi\omega}.$$

(Strictly speaking, we should treat the case $\omega = 0$ separately. However, as you can check, the formula that we obtained for $A(\omega)$ is valid in the limit as $\omega \to 0$.) Since $f(x)$ is even, $B(\omega) = 0$. For $|x| \neq 1$ the function is continuous and Theorem 1 gives

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin \omega \cos \omega x}{\omega}\, d\omega .$$

For $x = \pm 1$, points of discontinuity of f, Theorem 1 yields the value 1/2 for the last integral. Thus we have the Fourier integral representation of f

$$\frac{2}{\pi} \int_0^\infty \frac{\sin \omega \cos \omega x}{\omega}\, d\omega = \begin{cases} 1 & \text{if } |x| < 1, \\ 1/2 & \text{if } |x| = 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

■

In Example 1 we have used the evenness of f to infer that $B(\omega) = 0$ for all ω. Similarly, if f is odd, then $A(\omega) = 0$ for all ω. These observations simplify the computation of the Fourier integral representations of even and odd functions.

Theorem 1 can be used to evaluate interesting improper integrals. For example, setting $x = 0$ in the integral representation of Example 1 yields the important integral

(5) $$\int_0^\infty \frac{\sin \omega}{\omega}\, d\omega = \frac{\pi}{2},$$

known as the **Dirichlet integral**, after the German mathematician Peter Gustave Lejeune Dirichlet (1805–1859).

EXAMPLE 2 Computing integrals via the Fourier integral

Show that

$$\frac{2}{\pi} \int_0^\infty \frac{\cos \frac{\pi\omega}{2}}{1-\omega^2} \cos \omega x\, d\omega = \begin{cases} \cos x & \text{if } |x| < \pi/2, \\ 0 & \text{if } |x| > \pi/2. \end{cases}$$

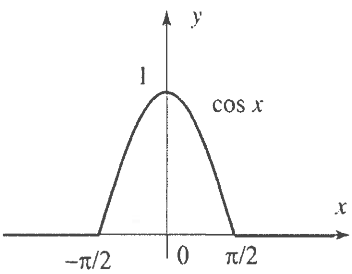


Figure 1 $f(x)$ in Example 2.

Solution Let $f(x)$ denote the function defined on the right side of this equality, as shown in Figure 1. It is even and vanishes outside the interval $[-\pi/2, \pi/2]$. Thus $B(\omega) = 0$ for all $\omega \geq 0$, and

$$\begin{aligned} A(\omega) &= \frac{2}{\pi} \int_0^{\pi/2} \cos x \cos \omega x\, dx && \text{(by evenness)} \\ &= \frac{1}{\pi} \int_0^{\pi/2} [\cos(1+\omega)x + \cos(1-\omega)x]\, dx && \text{(Trig identity)} \\ &= \frac{1}{\pi} \left[\frac{\sin[(1+\omega)\pi/2]}{1+\omega} + \frac{\sin[(1-\omega)\pi/2]}{1-\omega} \right] && (\text{for } \omega \neq 1) \\ &= \frac{1}{\pi} \left[\frac{\cos(\omega\pi/2)}{1+\omega} + \frac{\cos(\omega\pi/2)}{1-\omega} \right] = \frac{2\cos(\omega\pi/2)}{\pi(1-\omega^2)} . \end{aligned}$$

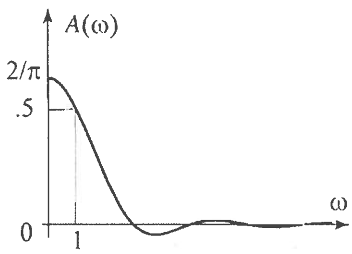


Figure 2 $A(\omega)$ is continuous.

The case $\omega = 1$ should be treated separately. It yields $A(1) = 1/2$ (check it!). As Figure 2 shows, the graph of $A(\omega)$ is continuous at $\omega = 1$. In fact, you can check that $\lim_{\omega\to 1} A(\omega) = A(1) = 1/2$. Now using (2), we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos(\omega\pi/2)}{1-\omega^2}\cos\omega x\, d\omega\,.$$

Replacing $f(x)$ by its formula, we get the desired identity. ■

Partial Fourier Integrals and the Gibbs Phenomenon

In analogy with the partial sums of Fourier series, we define the partial Fourier integral of f by

$$(6)\qquad S_\nu(x) = \int_0^\nu [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega \quad (\text{for } \nu > 0),$$

where $A(\omega)$ and $B(\omega)$ are given by (3). With this notation, Theorem 1 states

$$\lim_{\nu\to\infty} S_\nu(x) = \frac{f(x+) + f(x-)}{2}\,.$$

Like Fourier series, near a point of discontinuity the Fourier integral exhibits a Gibbs phenomenon. To illustrate this phenomenon, we introduce the **sine integral function**

$$(7)\qquad \operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt \qquad (-\infty < x < \infty).$$

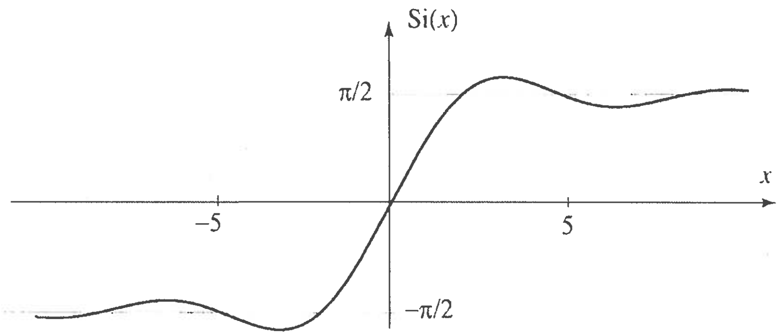


Figure 3 Graph of $\operatorname{Si}(x)$ Even though there is no expression of $\operatorname{Si}(x)$ in terms of elementary functions, you can still compute its numerical values using a power series expansion (see Exercise 23(c)).

Because of its frequent occurrence, the function $\operatorname{Si}(x)$ is tabulated and is available as a standard function in most computer systems. See Figure 3 for its graph. From (5), it follows that

$$(8)\qquad \lim_{x\to\infty} \operatorname{Si}(x) = \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}\,.$$

EXAMPLE 3 Gibbs phenomenon for partial Fourier integrals

(a) Show that the partial Fourier integral of the function in Example 1 can be written as

$$S_\nu(x) = \frac{1}{\pi}[\mathrm{Si}(\nu(1+x)) + \mathrm{Si}(\nu(1-x))].$$

(b) To illustrate the representation of the function by its Fourier integral, plot several partial Fourier integrals and discuss their behavior near the points $x = \pm 1$.

Solution We have from Example 1 and (6)

$$\begin{aligned} S_\nu(x) &= \frac{2}{\pi}\int_0^\nu \frac{\sin\omega\cos\omega x}{\omega}\,d\omega \\ &= \frac{1}{\pi}\int_0^\nu \frac{\sin\omega(1+x)}{\omega}\,d\omega + \frac{1}{\pi}\int_0^\nu \frac{\sin\omega(1-x)}{\omega}\,d\omega \end{aligned}$$

(Let $u = \omega(1+x)$ in the first integral, and $u = \omega(1-x)$ in the second.)

$$\begin{aligned} &= \frac{1}{\pi}\left[\int_0^{\nu(1+x)} \frac{\sin u}{u}\,du + \int_0^{\nu(1-x)} \frac{\sin u}{u}\,du\right] \\ &= \frac{1}{\pi}[\mathrm{Si}(\nu(1+x)) + \mathrm{Si}(\nu(1-x))], \qquad \text{by (7).} \end{aligned}$$

(b) In Figure 4 we have plotted the graphs of $S_\nu(x)$, for $\nu = 1, 4, 7, 10$, using the sine integral function and the formula given by (a). Observe how the partial integrals approximate the function in a way reminiscent of the approximation of a periodic function by its Fourier series. In particular, note the Gibbs phenomenon at the points of discontinuity of f where the partial integrals overshoot their limiting values. ■

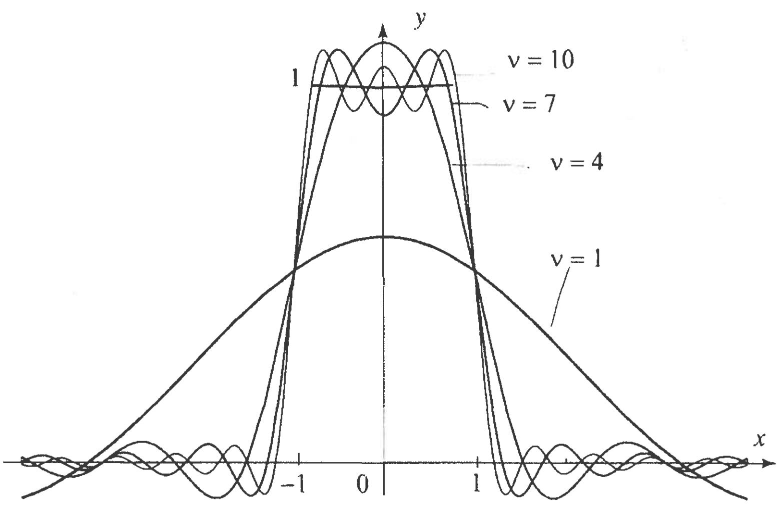


Figure 4 Approximation by partial Fourier integrals and Gibbs phenomenon.

Exercises 7.1

In Exercises 1–12, find the Fourier integral representation of the given function.

1.

$$f(x) = \begin{cases} 1 & \text{if } -a < x < a, \\ 0 & \text{otherwise,} \end{cases}$$

where $a > 0$.

2.

$$f(x) = \begin{cases} -1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

3.

$$f(x) = \begin{cases} 1 - \cos x & \text{if } -\pi/2 < x < \pi/2, \\ 0 & \text{otherwise.} \end{cases}$$

4.

$$f(x) = \begin{cases} 1 - |x| & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

5.

$$f(x) = e^{-|x|}.$$

6.

$$f(x) = \begin{cases} 1 - x^2 & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

7.

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

8.

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 1, \\ 1 & \text{if } 1 < |x| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

9.

$$f(x) = \begin{cases} x & \text{if } -1 < x < 1, \\ 2 - x & \text{if } 1 < x < 2, \\ -2 - x & \text{if } -2 < x < -1, \\ 0 & \text{otherwise.} \end{cases}$$

10.

$$f(x) = \begin{cases} x & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

11.

$$f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

12.

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

13. (a) Use Example 1 to show that

$$\int_0^\infty \frac{\sin \omega \cos \omega}{\omega}\, d\omega = \frac{\pi}{4}.$$

(b) Use integration by parts and (a) to obtain

$$\int_0^\infty \frac{\sin^2 \omega}{\omega^2}\, d\omega = \frac{\pi}{2}.$$

14. Use the identity $\sin^2 \omega + \cos^2 \omega = 1$ and Exercise 13(b) to obtain

$$\int_0^\infty \frac{\sin^4 \omega}{\omega^2}\, d\omega = \frac{\pi}{4}.$$

[Hint: $\sin^2 \omega = \sin^4 \omega + \cos^2 \omega \sin^2 \omega = \sin^4 \omega + \frac{1}{4} \sin^2 2\omega$.]

7.2 The Fourier Transform

We will use the complex exponential function to write the Fourier integral representation of Section 7.1 in complex form. This new representation features an important pair of transforms: the Fourier transform and its inverse Fourier transform. As you will see in this section, the concept of transform pairs provides a convenient way to state the fundamental operational properties of the Fourier transform, which are very useful in solving boundary value problems.

Consider a continuous piecewise smooth integrable function f. Starting with the Fourier integral representation, we have

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\,(\cos\omega t\cos\omega x + \sin\omega t\sin\omega x)\,dt\,d\omega \\ &\qquad (\cos(a-b) = \cos a\cos b + \sin a\sin b) \\ &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\omega(x-t)\,dt\,d\omega \\ &\qquad (\cos u = \frac{1}{2}(e^{iu} + e^{-iu})) \\ &= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(t)(e^{i\omega(x-t)} + e^{-i\omega(x-t)})\,dt\,d\omega \\ &= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(t)e^{i\omega(x-t)}\,dt\,d\omega + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(t)e^{-i\omega(x-t)}\,dt\,d\omega. \end{aligned}$$

If we change ω to $-\omega$ in the second term and adjust the limits on ω from $-\infty$ to 0, we obtain, after adding the two integrals,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)e^{i\omega(x-t)}\,dt\,d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{i\omega x} \overbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t)e^{-i\omega t}\,dt}^{\widehat{f}(\omega)}\,d\omega. \end{aligned}$$

This is the **complex form of the Fourier integral representation**, which features the following transform pair:

FOURIER TRANSFORM

(1) $$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{-i\omega x}dx \qquad (-\infty < \omega < \infty)$$

and

INVERSE FOURIER TRANSFORM

(2) $$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{i\omega x}\widehat{f}(\omega)\,d\omega \qquad (-\infty < x < \infty).$$

There are other conventions for the Fourier transform. For example, we could choose $\widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$, and then the inverse Fourier transform becomes $f(x) = \int_{-\infty}^{\infty} e^{i\omega x}\widehat{f}(\omega)\,d\omega$. In the definition of $\widehat{f}$ we have used x as a variable of integration, instead of t. The symbols $\mathcal{F}(f)(\omega)$ and $\mathcal{F}^{-1}(f)(x)$ are also used to denote the Fourier transform and its inverse, respectively. Sometimes, to be more specific, we will write $\mathcal{F}(f(x))(\omega)$ instead of $\mathcal{F}(f)(\omega)$. According to Theorem 1 of Section 7.1, if f is not continuous at x, the left side of (2) is to be replaced by $(f(x+) + f(x-))/2$. The integral for the inverse Fourier transform may not exist as a two-sided improper integral; in general, this integral should be computed as a Cauchy principal value: $f(x) = \frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^{a} e^{i\omega x}\widehat{f}(\omega)\,d\omega$ (see [1], Section 11.1).

Putting $\omega = 0$ in (1), we find that

$$\boxed{\widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,dx\,.}$$

Thus the value of the Fourier transform at $\omega = 0$ is equal to the signed area between the graph of $f(x)$ and the x-axis, multiplied by a factor of $1/\sqrt{2\pi}$.

EXAMPLE 1 A Fourier transform

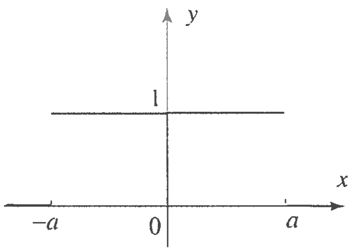


Figure 1 Graph of f in Example 1.

(a) Find the Fourier transform of the function in Figure 1, given by

$$f(x) = \begin{cases} 1 & \text{if } |x| < a, \\ 0 & \text{if } |x| > a. \end{cases}$$

What is $\widehat{f}(0)$? (b) Express f as an inverse Fourier transform.

Solution For $\omega \neq 0$ we have

$$\begin{aligned} \widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-i\omega x}\,dx \\ &= \frac{-1}{\sqrt{2\pi}i\omega}e^{-i\omega x}\Big|_{-a}^{a} = \sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}. \end{aligned}$$

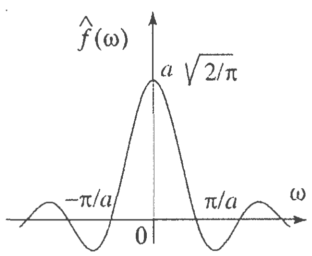


Figure 2 Graph of $\widehat{f}$ in Example 1.

For $\omega = 0$ we have $\widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} dx = a\sqrt{2/\pi}$. Since

$$\lim_{\omega\to 0}\widehat{f}(\omega) = \lim_{\omega\to 0}\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega} = a\sqrt{\frac{2}{\pi}} = \widehat{f}(0),$$

it follows that $\widehat{f}(\omega)$ is continuous at 0 (Figure 2), and we may write

$$\widehat{f}(\omega) = \sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega} \qquad \text{for all } \omega.$$

(b) To express f as an inverse Fourier transform, we use (2) and get

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x}\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}\,d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\omega x}\frac{\sin a\omega}{\omega}\,d\omega \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}(\cos\omega x - i\sin\omega x)\frac{\sin a\omega}{\omega}\,d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\omega x\sin a\omega}{\omega}\,d\omega, \end{aligned}$$

because $\sin\omega x\,\frac{\sin a\omega}{\omega}$ is an odd function of ω and so its integral is zero.

Not surprisingly, when $a = 1$, this representation coincides with the integral representation we found in Example 1 of Section 7.1. ■

The Fourier transform in Example 1 is continuous on the entire real line even though the function has jump discontinuities at $x = \pm a$. In fact, it can be shown that the Fourier transform of an integrable function is *always continuous*.

In our next example we will use the absolute value of complex numbers. Let us recall that if $z = a + ib$, then $|z| = \sqrt{a^2 + b^2}$. In particular, if $z = e^{-ix\omega}$, then

$$\left|e^{-ix\omega}\right| = |\cos(x\omega) - i\sin(x\omega)| = \sqrt{\cos^2(x\omega) + \sin^2(x\omega)} = 1\,.$$

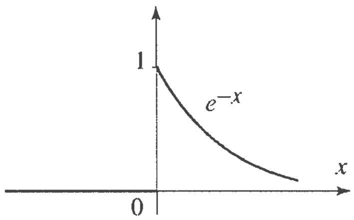


Figure 3 Graph of f in Example 2.

EXAMPLE 2 Computing Fourier transforms

Find the Fourier transform of the function in Figure 3,

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0\,. \end{cases}$$

Solution We have

$$\begin{aligned} \widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-x}e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-x(1+i\omega)}\,dx \\ &= \frac{-1}{\sqrt{2\pi}(1+i\omega)}e^{-i\omega x}e^{-x}\Big|_0^{\infty}\,. \end{aligned}$$

Since $\left|e^{-ix\omega}\right| = 1$, it follows that $\lim_{x\to\infty}\left|e^{-x(1+i\omega)}\right| = \lim_{x\to\infty} e^{-x} = 0$, and so

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}(1+i\omega)} = \frac{1-i\omega}{\sqrt{2\pi}(1+\omega^2)}\,.$$

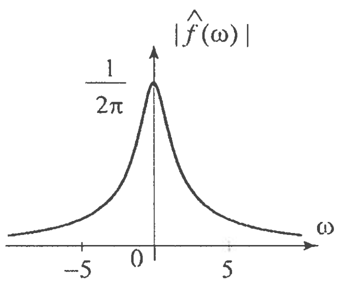


Figure 4 Graph of $|\widehat{f}|$ in Example 2.

Figure 4 shows the graph of the absolute value of $\widehat{f}$. Here again, it is worth noting that $\widehat{f}$ and $|\widehat{f}|$ are both continuous even though f is not. ■

Example 2 illustrates a noteworthy fact that the Fourier transform may be complex-valued even though the function is real-valued. Also, the Fourier transform is continuous but not integrable, not even as an improper two-sided integral. In this case, the integral in (2) for the inverse Fourier transform should be computed as a Cauchy principal value. Indeed, you can check that, at $x = 0$, we do have $1/2 = \big(f(0+) + f(0-)\big)/2$, and the Cauchy principal value of the inverse Fourier transform yields

$$\begin{aligned}\lim_{a\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\widehat{f}(\omega)\,d\omega &= \lim_{a\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\frac{1-i\omega}{\sqrt{2\pi}(1+\omega^2)}\,d\omega \\ &= \frac{1}{2\pi}\lim_{a\to\infty}\Big[\int_{-a}^{a}\frac{1}{1+\omega^2}\,d\omega - i\overbrace{\int_{-a}^{a}\frac{\omega}{1+\omega^2}\,d\omega}^{=0}\Big] = \frac{1}{2}.\end{aligned}$$

As defined by (1), the Fourier transform takes a function f and produces a new function $\widehat{f}$, and the inverse transform recovers the original function f from $\widehat{f}$. This process makes of transform pairs a powerful tool in solving partial differential equations. As we will see in the following sections, the idea is to "Fourier transform" a given equation into one that may be easier to solve. After solving the transformed equation involving $\widehat{f}$, we recover the solution of the original problem with the inverse transform. To assist us in handling the transformed equations, we develop the operational properties of the Fourier transform.

Operational Properties

We shall investigate the behavior of the Fourier transform in connection with the common operations on functions: linear combination, translation, dilation, differentiation, multiplication by polynomials, and convolution.

THEOREM 1 LINEARITY

The Fourier transform is a linear operation; that is, for any integrable functions f and g and any real numbers a and b,

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g).$$

Proof Exercise 18. ■

THEOREM 2 FOURIER TRANSFORMS OF DERIVATIVES

(i) Suppose $f(x)$ is piecewise smooth, $f(x)$ and $f'(x)$ are integrable, and $f(x) \to 0$ as $|x| \to \infty$, then

$$\mathcal{F}(f') = i\omega\,\mathcal{F}(f).$$

(ii) If in addition $f''(x)$ is integrable, and $f'(x)$ is piecewise smooth and $\to 0$ as $|x| \to \infty$, then

$$\mathcal{F}(f'') = i\omega\mathcal{F}(f') = -\omega^2\mathcal{F}(f).$$

(iii) In general, if f and $f^{(k)}(x)$ $(k = 1, 2, \ldots, n-1)$ are piecewise smooth and tend to 0 as $|x| \to \infty$, and f and its derivatives of order up to n are integrable, then

$$\mathcal{F}(f^{(n)}) = (i\omega)^n\mathcal{F}(f).$$

Proof Parts (ii) and (iii) are obtained by repeated applications of (i). To prove (i), we use the definition of $\mathcal{F}(f')$ and integrate by parts. To simplify the proof, we suppose further that f is smooth. Then

$$\begin{aligned}
\mathcal{F}(f')(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\omega x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left[f(x)\, e^{-i\omega x} \Big|_{-\infty}^{\infty} - (-i\omega) \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \right] \\
&= 0 + i\omega \mathcal{F}(f) \qquad (\text{since } f(x) \to 0 \text{ as } |x| \to \infty, \text{ and } |e^{\pm i\omega x}| = 1). \quad \blacksquare
\end{aligned}$$

THEOREM 3 DERIVATIVES OF FOURIER TRANSFORMS

(i) Suppose $f(x)$ and $xf(x)$ are integrable; then

$$\mathcal{F}(xf(x))(\omega) = i \left[\widehat{f}\right]'(\omega) = i \frac{d}{d\omega} \mathcal{F}(f)(\omega).$$

(ii) In general, if $f(x)$ and $x^n f(x)$ are integrable, then

$$\mathcal{F}(x^n f(x)) = i^n \left[\widehat{f}\right]^{(n)}(\omega).$$

Sketch of Proof Part (ii) follows from (i). To motivate (i) we will assume that we can differentiate under the integral. Then

$$\begin{aligned}
\left[\widehat{f}\right]'(\omega) &= \frac{d}{d\omega} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{d}{d\omega} e^{-i\omega x}\, dx \\
&= -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{-i\omega x}\, dx = -i\mathcal{F}(xf(x))(\omega),
\end{aligned}$$

and (i) follows upon multiplying both sides by i. This proof is valid if for example f is smooth and vanishes outside a finite interval. For an arbitrary function f, we can approximate f by functions that are smooth and vanish outside a finite interval. The details are beyond the level of this book and will be omitted. ■

Convolution of Functions

We expand our list of operational properties by introducing the convolution of two functions f and g by

CONVOLUTION

(3) $$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t) g(t)\, dt.$$

(The factor $\frac{1}{\sqrt{2\pi}}$ is merely for convenience. If we drop it from the definition of the convolution, it will reappear in its Fourier transform.) The convolution of f and g is a binary operation, which combines translation, multiplication of functions, and integration. Its effect on the functions f and g is difficult to explain directly, as the following examples illustrate. (It does have a simple description in terms of the Fourier transform, as we will see in Theorem 4.) Let us first observe that convolution is a commutative operation; that is, $f * g(x) = g * f(x)$. This follows by making a change of variables $(t \leftrightarrow t - x)$ in (3) (Exercise 55).

EXAMPLE 3 Convolution with the cosine

Suppose that f is integrable and even ($f(-x) = f(x)$ for all x) and let $g(x) = \cos ax$. Show that, for all real numbers a: $f * g(x) = \cos(ax)\,\widehat{f}(a)$.

Solution From the definition and the fact that $f * g = g * f$, we have

$$\begin{aligned} f * g(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos[a(x-t)]\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \big[\cos(ax)\cos(at) + \sin(ax)\sin(at) \big]\, dt. \end{aligned}$$

Since f is even, the product $f(t)\sin at$ is odd; hence $\int_{-\infty}^{\infty} f(t)\sin at\, dt = 0$, and so

$$\begin{aligned} f * g(x) &= \cos(ax)\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\cos(at)\, dt \\ &= \cos(ax)\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\big[\cos(at) - i\sin(at) \big]\, dt \\ &= \cos(ax)\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iat}\, dt = \cos(ax)\widehat{f}(a). \end{aligned}$$ ■

EXAMPLE 4 Convolution as an average

For $n = 1, 2, \ldots$, let $g_n(x) = n\sqrt{\pi/2}$ if $|x| < 1/n$ and $g_n(x) = 0$ otherwise. Suppose that f is continuous on $(-\infty, \infty)$ and let F denote an antiderivative of f. Show that

$$f * g_n(x) = \frac{F(x + 1/n) - F(x - 1/n)}{2/n}.$$

Solution From the definition, we have

$$\begin{aligned} f * g_n(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t) g_n(t)\, dt = \frac{n}{2} \int_{-1/n}^{1/n} f(x-t)\, dt \\ &= \frac{n}{2} \int_{x-1/n}^{x+1/n} f(t)\, dt = \frac{n}{2}\big(F(x+1/n) - F(x-1/n)\big), \end{aligned}$$

where we have used the change of variables $t \leftrightarrow x - t$ in the second integral. Thus the desired result follows. ■

We mention some noteworthy properties of the convolution in Example 4. The interval of integration, $(x - 1/n,\ x + 1/n)$, in the expression $f * g_n(x) = \frac{n}{2}\int_{x-1/n}^{x+1/n} f(t)\, dt$ is centered at x and has length $2/n$. Thus the convolution $f * g_n(x)$ is the average of the function f over the interval $(x - 1/n,\ x + 1/n)$. As the length of the interval shrinks to 0 (that is, as $n \to \infty$), we expect this average to converge to $f(x)$. In other terms, we expect $\lim_{n\to\infty} f * g_n(x) = f(x)$. This is indeed the case, since we have

$$\begin{aligned}\lim_{n\to\infty} f * g_n(x) &= \lim_{n\to\infty} \frac{n}{2}(F(x+1/n) - F(x-1/n))\\ &= \frac{1}{2}\lim_{1/n\to 0}\left[\frac{F(x+1/n)-F(x)}{1/n} + \frac{F(x+(-1/n))-F(x)}{(-1/n)}\right]\\ &= F'(x),\end{aligned}$$

by definition of the derivative. But $F'(x) = f(x)$, by the fundamental theorem of calculus; and so $\lim_{n\to\infty} f * g_n(x) = f(x)$.

The fact that the convolution of f with a sequence of functions converges to f will be at the heart of solutions of important boundary value problems, such as the Dirichlet problem in the upper half-plane and the heat equation of the real line. In each one of these problems, the sequence of functions, or kernels, will be different but it will share properties similar to the following properties of the sequence (g_n) in Example 4:

- $g_n(x) \geq 0$ for all x. The area under the graph of $g_n(x)$ and above the x-axis is equal to $\sqrt{2\pi}$, for all $n \geq 1$. That is, $\int_{-\infty}^{\infty} g_n(x)\,dx = \sqrt{2\pi}$. Moreover, the area is more and more concentrated around 0. That is, $\lim_{n\to\infty} \int_a^b g_n(x)\,dx = 0$ if 0 is not in $[a, b]$.
- On the Fourier transform side, we have $\lim_{n\to\infty} \widehat{g}_n(\omega) = 1$ for all ω (Exercise 8).

Convolutions can be tedious to compute directly from definition (3). One way to avoid a direct computation is to use the following important property of convolutions and the Fourier transform. The process is illustrated by Example 5 below.

THEOREM 4 FOURIER TRANSFORMS OF CONVOLUTIONS

Suppose that f and g are integrable; then

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g).$$

Theorem 4 is expressed by saying that the Fourier transform takes convolutions into products.

Proof Using (3) and (1), and then interchanging the order of integration, we get

$$\begin{aligned}\mathcal{F}(f*g)(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)e^{-i\omega x}\,dx\,g(t)\,dt\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}\,du\,e^{-i\omega t}g(t)\,dt\\ &\qquad (u = x-t,\ du = dx)\\ &= \mathcal{F}(f)(\omega)\mathcal{F}(g)(\omega).\end{aligned}$$

■

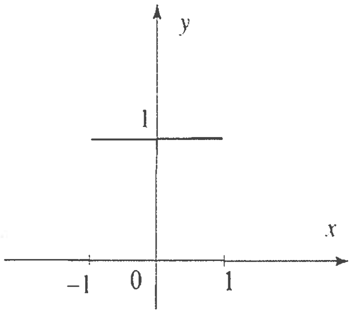


Figure 5 Graph of f.

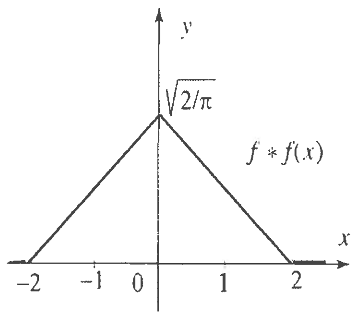


Figure 6 Graph of $f * f$.

EXAMPLE 5 Fourier transform of a convolution

Consider the function $f(x) = 1$ if $|x| < 1$ and 0 otherwise. The graph of this function is shown in Figure 5. From Example 1, we have

$$\widehat{f}(\omega) = \sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}.$$

We want to compute $f * f$, the convolution of f with itself. Instead of computing directly from (3), we will use Theorem 4 as follows. We have

$$\mathcal{F}(f * f)(\omega) = \widehat{f}(\omega)^2 = \frac{2}{\pi}\frac{\sin^2\omega}{\omega^2}.$$

Using the inverse Fourier transform, with the help of the table of Fourier transforms in Appendix B, we find

$$f * f(x) = \mathcal{F}^{-1}\left(\frac{2}{\pi}\frac{\sin^2\omega}{\omega^2}\right) = \begin{cases} \sqrt{\frac{2}{\pi}}\left(1 - \frac{|x|}{2}\right) & \text{if } |x| < 2, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

The graph of $f * f(x)$ is shown in Figure 6. Note that $f * f$ is continuous even though f is not. ■

The rest of this section is devoted to studying the **Gaussian function** $f(x) = e^{-x^2}$ and its Fourier transform. This function plays a key role in the solution of the heat equation on the line (Section 7.4). We need the famous improper integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}. \tag{4}$$

We have computed this integral in Section 4.7, Exercise 35, in connection with the gamma function. Let us give here a more direct proof. We square the integral, use polar coordinates ($r^2 = x^2 + y^2$, $dx\,dy = r\,dr\,d\theta$), and get

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^{\infty}\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} d\theta = \pi, \end{aligned}$$

and (4) follows upon taking square roots.

THEOREM 5
TRANSFORM OF THE GAUSSIAN

Let $a > 0$. We have

$$\mathcal{F}\left(e^{-\frac{ax^2}{2}}\right)(\omega) = \frac{1}{\sqrt{a}} e^{-\frac{\omega^2}{2a}}.$$

Proof We give an indirect proof based on the operational properties of the Fourier transform. Let $f(x) = e^{-\frac{ax^2}{2}}$. A simple verification shows that f satisfies the first order linear differential equation

$$f'(x) + axf(x) = 0.$$

Taking Fourier transforms and using Theorems 1, 2, and 3, we get

$$\omega \widehat{f}(\omega) + a\frac{d}{d\omega}[\widehat{f}](\omega) = 0.$$

Thus $\widehat{f}$ satisfies a similar first order linear ordinary differential equation. Solving this equation in $\widehat{f}$, we find

$$\widehat{f}(\omega) = A\, e^{-\frac{\omega^2}{2a}},$$

where A is an arbitrary constant. To complete the proof, we must show that $A = \frac{1}{\sqrt{a}}$. We have

$$\begin{aligned} A &= \widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}}\,dx \\ &= \frac{1}{\sqrt{a\pi}}\int_{-\infty}^{\infty} e^{-u^2}\,du \qquad \left(u = \sqrt{\frac{a}{2}}x,\ \sqrt{\frac{2}{a}}\,du = dx\right) \\ &= \frac{1}{\sqrt{a}} \qquad \text{(by (4))}. \end{aligned}$$

■

Replacing a by $2a$ in Theorem 5 yields

(5) $$\mathcal{F}\left(e^{-ax^2}\right)(\omega) = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}} \qquad (a > 0).$$

Taking $a = 1$ in Theorem 5 gives

$$\mathcal{F}\left(e^{-\frac{x^2}{2}}\right)(\omega) = e^{-\frac{\omega^2}{2}}.$$

This remarkable identity states that $e^{-\frac{x^2}{2}}$ is its own Fourier transform. Is this the only function with this property? See Exercise 59 for an answer.

Theorem 5 can be used to compute some interesting integrals.

EXAMPLE 6 A special improper integral
Writing explicitly the Fourier transform in Theorem 5, we find that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}}(\cos\omega x - i\sin\omega x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}}\cos\omega x\,dx = \frac{1}{\sqrt{a}}e^{-\frac{\omega^2}{2a}}.$$

Taking $\omega = 1$ and $a = 2$, we get the interesting identity

$$\int_{-\infty}^{\infty} e^{-x^2}\cos x\,dx = \sqrt{\pi}\,e^{-\frac{1}{4}}.$$ ■

In the next section, we develop the Fourier transform method for solving partial differential equations on the real line. We will use this method to solve boundary value problems associated with the heat and wave equations, and a variety of other important problems on the real line and regions in the two dimensional space.

Exercises 7.2

In Exercises 1–7, (a) plot the given function and find its Fourier transform. (b) If $\widehat{f}$ is real-valued, plot it; otherwise plot $|\widehat{f}|$.

1.
$$f(x) = \begin{cases} -1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

2.
$$f(x) = \begin{cases} 1 - x^2 & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

3.
$$f(x) = \begin{cases} \sin x & \text{if } |x| \le \pi, \\ 0 & \text{otherwise.} \end{cases}$$

4.
$$f(x) = \begin{cases} x & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

5.
$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

6.
$$f(x) = \begin{cases} 1 - \frac{|x|}{a} & \text{if } |x| \le a, \\ 0 & \text{otherwise,} \end{cases}$$

where $a > 0$.

7.
$$f(x) = \begin{cases} 2x & \text{if } 0 < x < 10, \\ 0 & \text{otherwise.} \end{cases}$$

8. Let $g_n(x)$ be as in Example 4. Compute $\widehat{g}_n(\omega)$ and show that $\lim_{n\to\infty}\widehat{g}_n(\omega) = 1$ for all ω.

9. Compute $\widehat{f}(0)$ in Exercises 1 and 7 by looking at the graph of $f(x)$.

10. Reciprocity relation for the Fourier transform.
(a) From the definition of the transforms, explain why $\mathcal{F}(f)(x) = \mathcal{F}^{-1}(f)(-x)$.
(b) Use (a) to derive the **reciprocity relation** $\mathcal{F}^2(f)(x) = f(-x)$, where $\mathcal{F}^2(f) = \mathcal{F}(\mathcal{F}(f))$.
(c) Conclude the following: f is even if and only if $\mathcal{F}^2(f)(x) = f(x)$; f is odd if and only if $\mathcal{F}^2(f)(x) = -f(x)$.
(d) Show that for any f, $\mathcal{F}^4(f) = f$.