THE LAPLACE TRANSFORM

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Should I refuse a good dinner simply because I do not understand the process of digestion?

—OLIVER HEAVISIDE

[Criticized for using formal mathematical manipulations, without understanding how they worked.]

In the previous chapter we introduced the Fourier transform and the Fourier sine and cosine transforms and showed their utility in solving various boundary value problems for partial differential equations on unbounded domains. The problems to which these transforms applied were typically treated in Cartesian coordinates. Another transform that can frequently be applied with success is the Laplace transform, our first topic in this chapter. If one of the variables occurring in a problem ranges over a half-line $[0, \infty)$, we can often make progress by performing a Laplace transform with respect to this variable, in much the same way that we did with the sine and cosine transforms. Because of the importance of this transform in other settings, we present a self-contained treatment, including the solution of initial value problems for ordinary differential equations. For problems with other than Cartesian geometry, there are yet other transforms that are more natural and therefore more useful. For example, in unbounded problems with radial symmetry in either the plane or the space, so that the appropriate coordinates are polar, cylindrical, or spherical, the natural transform for the radial variable (r or ρ) involves Bessel functions. This transform, which depends on the order ν of the Bessel function involved, is known as the Hankel transform of order ν.

The Laplace Transform

In this section we present the definition and basic properties of the Laplace transform. As a warm-up for the applications with partial differential equations, we will use it to solve some simple ordinary differential equations.

Suppose that $f(t)$ is defined for all $t \geq 0$. The **Laplace transform** of f is the function

As a convention, functions $f, g, \ldots$ are defined for $t \geq 0$ and their transforms $F, G, \ldots$ are defined on the s-axis.

(1) $$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt\,.$$

Another commonly used notation for $\mathcal{L}(f)(s)$ is $F(s)$. For the integral to exist f cannot grow faster than an exponential. This motivates the following definition. We say that f is of **exponential order** if there exist positive numbers a and M such that

(2) $$|f(t)| \leq Me^{at} \quad \text{for all } t \geq 0\,.$$

For example, the functions 1, $4\cos 2t$, $5t\sin 2t$, e^{3t} are all of exponential order. We can now give a sufficient condition for the existence of the Laplace transform.

THEOREM 1 EXISTENCE OF THE LAPLACE TRANSFORM

Suppose that f is piecewise continuous on the interval $[0, \infty)$ and of exponential order with $|f(t)| \leq Me^{at}$ for all $t \geq 0$. Then $\mathcal{L}(f)(s)$ exists for all $s > a$.

Proof We have to show that for $s > a$

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt < \infty\,.$$

With M and a as above, we have

$$\begin{aligned}\left|\int_0^\infty f(t)e^{-st}\,dt\right| &\leq \int_0^\infty |f(t)|e^{-st}\,dt \leq M\int_0^\infty e^{at}e^{-st}\,dt \\ &= M\int_0^\infty e^{-(s-a)t}\,dt = \frac{M}{s-a} < \infty\,.\end{aligned}$$

■

Note that the function $\frac{1}{\sqrt{t}}$ is not of exponential order, because of its behavior at $t = 0$. However, we will show in Example 2 below that its Laplace transform $\mathcal{L}\left(\frac{1}{\sqrt{t}}\right)(s)$ exists for all $s > 0$. Thus Theorem 1 provides sufficient but not necessary conditions for the existence of the Laplace transform.

EXAMPLE 1 $\mathcal{L}(1)$, $\mathcal{L}(t)$, **and** $\mathcal{L}(e^{\alpha t})$

We compute these transforms using (1). We have

$$\mathcal{L}(1)(s) = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}, \quad s > 0;$$

$$\mathcal{L}(t)(s) = \int_0^\infty te^{-st}\,dt = \left(-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_0^\infty = \frac{1}{s^2}, \quad s > 0;$$

and finally, for $s > \alpha$,

$$\mathcal{L}(e^{\alpha t})(s) = \int_0^\infty e^{-(s-\alpha)t}\,dt = -\frac{1}{s-\alpha}e^{-(s-\alpha)t}\Big|_0^\infty = \frac{1}{s-\alpha}.$$

Note that $\mathcal{L}(e^{\alpha t})(s)$ is not defined for $s \le \alpha$. ■

In computing $\mathcal{L}(t)$ we had to integrate by parts once. Similarly, we could compute $\mathcal{L}(t^n)$ (n a positive integer) by integrating by parts n times. Rather than doing this, we shall take advantage of an interesting connection between the Laplace transform and the gamma function (Section 4.7).

EXAMPLE 2 $\mathcal{L}(t^a)$: **the gamma function**

(a) Evaluate $\mathcal{L}(t^a)(s)$ when $a > -1$ and $s > 0$.

(b) Derive from (a) the transforms $\mathcal{L}(t)$, $\mathcal{L}(t^2)$, and, more generally, $\mathcal{L}(t^n)$, where n is a positive integer.

(c) What is $\mathcal{L}\left(\frac{1}{\sqrt{t}}\right)$?

Solution (a) From (1) we have

$$\mathcal{L}(t^a)(s) = \int_0^\infty t^a e^{-st}\,dt.$$

To compare with the definition of the gamma function (13), Section 4.7, we make the change of variables $st = T$, $dt = \frac{1}{s}\,dT$. Then

$$\begin{aligned}\mathcal{L}(t^a)(s) &= \int_0^\infty \left(\frac{T}{s}\right)^a e^{-T}\frac{dT}{s} = \frac{1}{s^{a+1}}\underbrace{\int_0^\infty T^a e^{-T}\,dT}_{=\Gamma(a+1)} \\ &= \frac{\Gamma(a+1)}{s^{a+1}} \quad \text{(from (13), Section 4.7)}.\end{aligned}$$

(b) Using (a),

$$\begin{aligned}\mathcal{L}(t) &= \frac{\Gamma(2)}{s^2} = \frac{1}{s^2}; \\ \mathcal{L}(t^2) &= \frac{\Gamma(3)}{s^3} = \frac{2}{s^3};\end{aligned}$$

and, more generally,

$$\mathcal{L}(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}.$$

(c) Using (a), and (15), Section 4.7,

$$\mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = \mathcal{L}\left(t^{-1/2}\right) = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}. \qquad \blacksquare$$

Operational Properties

We will derive in the rest of this section properties of the Laplace transform that will assist us in solving differential equations. We are particularly interested in those formulas involving a function, its transform, and the transform of its derivatives. These formulas are similar to the operational properties of the Fourier transform. Because the Laplace transform is defined by an integral over the interval $[0, \infty)$, some of the formulas will involve the values of the function and its derivatives at 0.

THEOREM 2 LINEARITY

If f and g are functions and α and β are numbers, then

$$\mathcal{L}(\alpha f + \beta g) = \alpha\mathcal{L}(f) + \beta\mathcal{L}(g).$$

The proof is left as an exercise. You should also think about the domain of definition of $\mathcal{L}(\alpha f + \beta g)$ in terms of the domains of definition of $\mathcal{L}(f)$ and $\mathcal{L}(g)$.

EXAMPLE 3 $\mathcal{L}(\cos kt)$ **and** $\mathcal{L}(\sin kt)$

These transforms can be evaluated directly by using (1). Our derivation will be based on Euler's identity $e^{ikt} = \cos kt + i \sin kt$ and the linearity of the Laplace transform. We have

$$\begin{aligned}
\mathcal{L}(\cos kt) + i\mathcal{L}(\sin kt) &= \int_0^\infty (\cos kt + i \sin kt)e^{-st}\,dt \\
&= \int_0^\infty e^{-t(s-ik)}\,dt = -\frac{e^{-t(s-ik)}}{s-ik}\bigg|_0^\infty = \frac{1}{s-ik} \\
&= \frac{s+ik}{s^2+k^2} = \frac{s}{s^2+k^2} + i\,\frac{k}{s^2+k^2}.
\end{aligned}$$

Equating real and imaginary parts, we get

$$\mathcal{L}(\cos kt) = \frac{s}{s^2+k^2} \quad \text{and} \quad \mathcal{L}(\sin kt) = \frac{k}{s^2+k^2}.$$

For an alternative derivation, see Example 10 below. $\blacksquare$

The next result is very useful. It states that the Laplace transform takes derivatives into powers of s.

THEOREM 3 LAPLACE TRANSFORMS OF DERIVATIVES

(i) Suppose that f is continuous on $[0,\infty)$ and of exponential order as in (2). Suppose further that f' is piecewise continuous on $[0,\infty)$ and of exponential order. Then

(3) $$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)\,.$$

(ii) More generally, if $f, f', \ldots, f^{(n-1)}$ are continuous on $[0,\infty)$ and of exponential order as in (2), and $f^{(n)}$ is piecewise continuous on $[0,\infty)$ and of exponential order, then

(4) $$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)\,.$$

Proof Since f is of exponential order, then (2) holds for some positive constants a and M. The transform $\mathcal{L}(f')(s)$ is to be computed for $s > a$. Before we start the computation, note that for $s > a$

$$\lim_{t\to\infty} |f(t)|\,e^{-st} = \lim_{t\to\infty} \underbrace{|f(t)|}_{\leq Me^{at}}\, e^{-at}e^{-(s-a)t} \leq M \lim_{t\to\infty} e^{-(s-a)t} = 0,$$

because $s - a > 0$. We now compute, using (1) and integrating by parts,

$$\begin{aligned}
\mathcal{L}(f')(s) &= \int_0^\infty f'(t)e^{-st}\,dt \quad (s > a)\\
&= f(t)e^{-st}\Big|_0^\infty - (-s)\underbrace{\int_0^\infty f(t)e^{-st}\,dt}_{\mathcal{L}(f)(s)}\\
&= -f(0) + s\,\mathcal{L}(f),
\end{aligned}$$

which proves (i). Part (ii) follows by repeated applications of (i). ■

When $n = 2$, (4) gives

(5) $$\boxed{\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)\,.}$$

The following is a counterpart of Theorem 3 showing that the Laplace transform takes powers of t into derivatives.

THEOREM 4 DERIVATIVES OF TRANSFORMS

(i) Suppose $f(t)$ is piecewise continuous and of exponential order. Then

(6) $$\mathcal{L}(t\,f(t))(s) = -\frac{d}{ds}\,\mathcal{L}(f)(s)\,.$$

(ii) In general, if $f(t)$ is piecewise continuous and of exponential order, then

(7) $$\mathcal{L}(t^n f(t)) = (-1)^n\frac{d^n}{ds^n}\,\mathcal{L}(f)(s)\,.$$

Proof Differentiation under the integral sign gives

$$\begin{aligned}
[\mathcal{L}(f)]'(s) &= \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)\frac{d}{ds}e^{-st}\,dt \\
&= -\int_0^\infty t f(t)e^{-st}\,dt = -\mathcal{L}(tf(t))(s),
\end{aligned}$$

and (i) follows upon multiplying both sides by -1. Part (ii) is obtained by repeated applications of (i). ■

EXAMPLE 4 Derivatives of transforms

(a) Evaluate $\mathcal{L}(t\sin 2t)$. (b) Evaluate $\mathcal{L}(t^2\sin t)$.

Solution (a) Using (6) and Example 3, we find

$$\mathcal{L}(t\sin 2t) = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right] = \frac{4s}{(s^2+4)^2}.$$

(b) Similarly, using (7) and Example 3, we find

$$\mathcal{L}(t^2\sin t) = \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right] = \frac{2(-1+3s^2)}{(s^2+1)^3}.$$ ■

The following theorem states that multiplication of a function by $e^{\alpha t}$ causes the transform to be shifted by α units on the s-axis. This very important property has a counterpart that involves a shift on the t-axis (see Theorem 1, Section 8.2).

THEOREM 5 SHIFTING ON THE s-AXIS

Suppose that f is of exponential order. Let α be a real number and a be as in (2). For $s > a + \alpha$, we have

$$\mathcal{L}(e^{\alpha t}f(t))(s) = F(s-\alpha),$$

where $F(s) = \mathcal{L}(f(t))(s)$.

Proof Note that $e^{\alpha t}f(t)$ is also of exponential order and (2) holds with a replaced by $a+\alpha$. Thus Theorem 1 guarantees the existence of $\mathcal{L}(e^{\alpha t}f(t)$ for $s > a+\alpha$. We have

$$\mathcal{L}(e^{\alpha t}f(t))(s) = \int_0^\infty f(t)e^{\alpha t}e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-\alpha)t}\,dt = F(s-\alpha).$$ ■

By taking $f = 1$ in Theorem 5, we obtain the third transform in Example 1, $\mathcal{L}(e^{\alpha t}) = \frac{1}{s-\alpha}$, since $\mathcal{L}(1) = \frac{1}{s}$.

The Inverse Laplace Transform

We now reverse the process of computing Laplace transforms and instead, for a given function $F(s)$, we look for the function $f(t)$ whose Laplace transform is $F(s)$. The function $f(t)$ is called the **inverse Laplace transform** of $F(s)$ and is denoted by

$$f(t) = \mathcal{L}^{-1}(F(s)) \quad \text{or simply} \quad f = \mathcal{L}^{-1}(F).$$

It is possible to give a formula for the inverse Laplace transform like the one we gave for the inverse Fourier transform. The formula involves integration in the complex plane and is not very useful for our purposes (see [1], Chapter 12). Instead, we will compute the inverse transform by using known Laplace transforms, as illustrated by the examples below. We note that the inverse of any linear transform is itself linear. In particular, we have

$$\boxed{\mathcal{L}^{-1}(\alpha F + \beta G) = \alpha \mathcal{L}^{-1}(F) + \beta \mathcal{L}^{-1}(G).}$$

EXAMPLE 5 Inverse Laplace transforms

(a) Evaluate $\mathcal{L}^{-1}(\frac{2}{4+(s-1)^2})$. (b) Evaluate $\mathcal{L}^{-1}(\frac{1}{s^2+2s+3})$.

Solution (a) From the table of Laplace transforms in Appendix B (or by using Example 3 and Theorem 5), we find that

$$\mathcal{L}(e^{at}\sin kt) = \frac{k}{(s-a)^2+k^2}.$$

Taking $a = 1$ and $k = 2$, we get

$$\mathcal{L}(e^t \sin 2t) = \frac{2}{(s-1)^2+4}.$$

Hence

$$\mathcal{L}^{-1}(\frac{2}{4+(s-1)^2}) = e^t \sin 2t.$$

(b) Motivated by part (a), we first write

$$\frac{1}{s^2+2s+3} = \frac{1}{(s+1)^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}.$$

Now using the transform in (a) with $a = -1$ and $k = \sqrt{2}$, we get

$$\mathcal{L}^{-1}(\frac{1}{s^2+2s+3}) = \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left(\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right) = \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t. \qquad \blacksquare$$

"breuk splitsen"

Rational Functions and Partial Fractions

We just computed in Example 5 the inverse Laplace transforms of two rational functions. For more general rational functions, we can use a technique that is common for computing integrals of such functions. This technique is based on the fact that any rational function has a **partial fractions decomposition**, which expresses the function as a sum of a polynomial plus a finite number of rational functions of a simpler form. We will not prove this fact; instead, we will describe this decomposition, and show by examples how it works.

It is a fact from algebra that any polynomial $q(x)$ can be factored as a product of linear terms and quadratic polynomials with no real roots:

$$(8)\quad q(x) = a(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x^2+b_1x+c_1)^{n_1}(x^2+b_2x+c_2)^{n_2}\cdots,$$

where we have only finitely many factors, m_j and n_k are integers ≥ 1, all the a_j's are distinct, and all the quadratic factors are distinct and have no real roots. (This factorization follows from the fundamental theorem of algebra that states that every polynomial factors as a product of linear factors over the complex numbers. If the polynomial has real coefficients, its complex roots come in conjugate pairs. Now if a is a complex root with a nonzero imaginary part, then $(x-a)(x-\bar{a}) = x^2 - 2\,\mathrm{Re}\,(a) + |a|^2$ is a quadratic polynomial with no real roots. Thus if we multiply pairwise the linear factors that correspond to conjugate roots, we obtain the factorization (8) where all the coefficients are real.)

Given a rational function $p(x)/q(x)$, where p and q are polynomials with real coefficients, factor p and q as in (8), and cancel the common factors. So we may assume that p and q have no common factors. Without loss of generality, assume that the degree of p is strictly smaller than that of q; otherwise, divide p by q and write the rational function as the sum of a polynomial plus a rational function in which the degree of the numerator is strictly smaller than that of the denominator. Using the factorization (8) of q, you generate the partial fraction decomposition of p/q as follows. For each linear factor $(x-a)^m$ $(m \geq 1)$ that appears in (8), add the terms

$$(9)\qquad \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m},$$

where $A_1, \ldots, A_m$ are real numbers to be determined. And for each quadratic factor $(x^2+bx+c)^n$ $(n \geq 1)$ that appears in (8), add the terms

$$(10)\qquad \frac{B_1x+C_1}{(x^2+bx+c)} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(x^2+bx+c)^n},$$

where $B_1, \ldots, B_n$, $C_1, \ldots, C_n$ are real numbers to be determined. The partial fractions decomposition of $p(x)/q(x)$ is the sum of all the terms of the form (9) and (10). That is, $p(x)/q(x)$ is the equal to the sum of all these terms. The coefficients A_j, B_k, and C_k, can be computed by recombining the partial fractions and comparing with the coefficients of p, or by inspection, as we now illustrate.

EXAMPLE 6 Partial fractions decomposition

Find the partial fractions decompositions of the following rational functions:

(a) $\dfrac{1}{(x^2+1)(x+1)^2}$.   (b) $\dfrac{4x}{(x^2+1)^2(x-1)}$.

Solution (a) The partial fractions decomposition is of the form

(11) $$\frac{1}{(x^2+1)(x+1)^2} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{Bx+C}{x^2+1}.$$

Multiplying both sides by $(x+1)^2$, then evaluating at $x=-1$, we find $\frac{1}{2} = A_2$. Multiplying both sides by $(x+1)^2$, differentiating with respect to x, and then evaluating at $x=-1$, we find

$$\left.\frac{-2x}{(x^2+1)^2}\right|_{x=-1} = A_1 + \overbrace{\left.\frac{d}{dx}\frac{Bx+C}{x^2+1}(x+1)^2\right|_{x=-1}}^{=0} \quad\Rightarrow\quad \frac{1}{2} = A_1.$$

Substituting A_1 and A_2 by their values in the partial fractions decomposition and then evaluating at $x=0$, we get

$$1 = \frac{1}{2} + \frac{1}{2} + C \quad\Rightarrow\quad C = 0.$$

Multiplying both sides of (11) by x, then letting x tend to ∞, we find

$$0 = \lim_{x\to\infty}\left[\frac{x}{2(x+1)} + \frac{x}{2(x+1)^2} + \frac{Bx^2}{x^2+1}\right] = \frac{1}{2} + B.$$

Hence $B = -\frac{1}{2}$, and so

$$\frac{1}{(x^2+1)(x+1)^2} = \frac{1}{2(x+1)} + \frac{1}{2(x+1)^2} - \frac{x}{2(x^2+1)}.$$

(b) Write

(12) $$\frac{4x}{(x^2+1)^2(x-1)} = \frac{A}{x-1} + \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2}.$$

We multiply by $(x-1)$ then set $x=1$, and get $A=1$. Multiplying by $(x^2+1)^2$ and then setting $x=\pm i$ (where $i^2=-1$), we get

$$\frac{4i}{i-1} = B_2 i + C_2 \qquad (\text{for } x = i),$$

$$\frac{4(-i)}{-i-1} = B_2(-i) + C_2 \qquad (\text{for } x = -i).$$

Solving the equations, we find $B_2 = -2$ and $C_2 = 2$. Setting $x = 0$, we find

$$0 = -A + C_1 + C_2 \Rightarrow C_1 = A - C_2 = -1.$$

Finally, multiplying both sides of (12) by x, then letting $x \to \infty$, we find

$$0 = A + B_1 \quad \Rightarrow \quad B_1 = -1.$$

So

$$\frac{4x}{(x^2+1)^2(x-1)} = \frac{1}{x-1} - \frac{x+1}{x^2+1} + \frac{-2x+2}{(x^2+1)^2}.$$

■

We next compute inverse Laplace transforms using partial fractions.

EXAMPLE 7 Inverse Laplace transform of rational functions
Evaluate $\mathcal{L}^{-1}\left(\frac{1}{s^2+2s-3}\right)$.

Solution (First Method) As we did in Example 5(b), start by writing

$$\frac{1}{s^2+2s-3} = \frac{1}{(s+1)^2-2^2} = \frac{1}{2}\frac{2}{(s+1)^2-2^2}.$$

From the table of Laplace transforms in Appendix B, we have

$$\mathcal{L}(e^{at}\sinh kt) = \frac{k}{(s-a)^2-k^2}.$$

Taking $a = -1$ and $k = 2$, it follows that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s-3}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2-2^2}\right) = \frac{1}{2}e^{-t}\sinh 2t.$$

(Second Method) Here we use partial fractions. First factor the denominator as $s^2+2s-3 = (s+3)(s-1)$. Now write

$$\frac{1}{(s+3)(s-1)} = \frac{A}{(s+3)} + \frac{B}{(s-1)}.$$

Multiply both sides by $(s+3)(s-1)$, then

$$1 = A(s-1) + B(s+3).$$

Setting $s = 1$ and then $s = -3$ yields $B = \frac{1}{4}$ and $A = -\frac{1}{4}$, respectively. Thus

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s-3}\right) = -\frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = -\frac{1}{4}e^{-3t} + \frac{1}{4}e^{t}.$$

It is easy to see that this transform is also equal to $\frac{1}{2}e^{-t}\sinh 2t$, matching our earlier finding. ■

Laplace Transform and Ordinary Differential Equations

The key to solving differential equations via the Laplace transform method is to use the operational properties, particularly those related to differentiation.

We begin with a simple initial value problem. In what follows, we will denote the Laplace transform of $y(t)$ by $Y(s)$.

EXAMPLE 8 A second order ordinary differential equation
Solve $y'' + y = 2, \quad y(0) = 0,\ y'(0) = 1.$

Solution Taking the Laplace transform of both sides of the equation and using Theorem 3, we find

$$s^2Y - sy(0) - y'(0) + Y = \mathcal{L}(2) = \frac{2}{s}.$$

Using the initial conditions, we obtain

$$(s^2 + 1)Y - 1 = \frac{2}{s} \quad \Rightarrow \quad Y = \frac{1}{s^2+1} + \frac{2}{s(s^2+1)}.$$

Using partial fractions on the second term, we find

$$Y = \frac{1}{s^2+1} + \frac{2}{s} - \frac{2s}{s^2+1}.$$

Finally, taking the inverse Laplace transform, we get $y = \sin t + 2 - 2\cos t$. ■

This example is a typical illustration of the Laplace transform method. Starting from a linear ordinary differential equation with constant coefficients in y, the Laplace transform produces an algebraic equation that can be solved for Y. The solution y is then found by taking the inverse Laplace transform of Y. The Laplace transform method is most compatible with initial value problems where the initial data is given at $t = 0$, owing to the way the transform acts on derivatives. If the initial data is given at some other value t_0, the Laplace transform still applies: We simply make the change of variables $\tau = t - t_0$. The next example illustrates this process.

EXAMPLE 9 Shifting the time variable
Solve $y'' + 2y' + y = t, \quad y(1) = 0,\ y'(1) = 0.$

Solution Making the change of variables $\tau = t - 1$, we arrive at the initial value problem

$$y'' + 2y' + y = \tau + 1, \quad y(0) = 0,\ y'(0) = 0,$$

where now a prime denotes differentiation with respect to τ. From this point, we proceed as in Example 7. Transforming yields

$$s^2Y + 2sY + Y = \frac{1}{s^2} + \frac{1}{s} \quad \Rightarrow \quad Y = \frac{1}{s^2(s+1)}.$$

Using partial fractions, we get

$$Y = \frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}.$$

LT-Exercises

show that the given function is of exponential order by establishing (2) with an appropriate choice of the numbers a and M.

LT1 $f(t) = 11 \cos 3t$.

LT2 $f(t) = 5 e^{3t}$.

LT3 $f(t) = \sinh 3t$.

* evaluate the Laplace transform of the given function using appropriate theorems and examples from this section.

LT4 $f(t) = 2t + 3$.

LT5 $f(t) = \sqrt{t} + \frac{1}{\sqrt{t}}$.

LT6 $f(t) = t^2 e^{3t}$.

LT7 $f(t) = t \sin 4t$.

* evaluate the inverse Laplace transform of the given function.

LT8 $F(s) = \frac{1}{s^2}$.

LT9 $F(s) = \frac{4}{3s^2+1}$.

LT10 $F(s) = \frac{1}{(s-3)^5} + \frac{s-3}{1+(s-3)^2}$.

* solve the given initial value problem with the Laplace transform.

LT11 $y' + y = \cos 2t, \quad y(0) = -2$.

LT12 $y'' + y = \cos t, \quad y(\pi) = 0, \ y'(\pi) = 0$.

LT13 $y'' + 2y' + y = te^{-2t}, \quad y(0) = 1, \ y'(0) = 1$.

LT14 $y'' - y' - 6y = e^t \cos t, \quad y(0) = 0, \ y'(0) = 1$.