1)

$$\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

$\mathcal{L}$-trafo :
$$s^2 Y(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0} + 2(sY(s) - \overbrace{y(0)}^{=1}) + 5Y(s) = 0$$

$$\Rightarrow \left(s^2 Y(s) - s\right) + 2(sY(s) - 1) + 5Y(s) = 0$$

$$\Rightarrow (s^2 + 2s + 5)Y(s) = s + 2$$

$$Y(s) = \frac{s+2}{s^2+2s+5}$$

$$= \frac{s+2}{(s+1)^2+4}$$

$$= \frac{s+1 \; +1}{(s+1)^2+4}$$

$$= \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}$$

$$= \frac{s+1}{(s+1)^2+4} + \frac{1}{2}\cdot\frac{2}{(s+1)^2+4}$$

$\mathcal{L}^{-1}$:
$$y(t) = e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)$$

formula sheet Laplace transf.

$a = -1$
$b = 2$

2) Consider the second order DE:

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$, $y'(0) = 2$

taking the Laplace transform gives:

$$\mathcal{L}(y'' - 10y' + 9y) = \mathcal{L}(5t) = 5\mathcal{L}(t) = \frac{5}{s^2}$$

$$\frac{5}{s^2} = \mathcal{L}(y'') - 10\mathcal{L}(y') + 9\mathcal{L}(y)$$

using Theorem 3 we get:

$$\frac{5}{s^2} = s^2\mathcal{L}(y) - s\,y(0) - y'(0) - 10\left(s\mathcal{L}(y) - y(0)\right)$$
$$+ 9\mathcal{L}(y)$$

Rearranging gives:

$$\frac{5}{s^2} = (s^2 - 10s + 9)\mathcal{L}(y) + (10 - s)y(0) - y'(0)$$

$$= (s^2 - 10s + 9)\mathcal{L}(y) + s - 10 - 2$$

solving for $\mathcal{L}(y)$ gives:

$$\mathcal{L}(y) = \frac{\frac{5}{s^2} - s + 12}{s^2 - 10s + 9} \cdot \frac{s^2}{s^2} = \frac{5 - s^3 + 12s^2}{(s^2 - 10s + 9)s^2}$$

$$= \frac{-s^3 + 12s^2 + 5}{(s-9)(s-1)s^2}$$

using partial fractions this becomes equal to ("breuksplitsen")

$$\mathcal{L}(y) = \frac{A}{s-9} + \frac{B}{s-1} + \frac{C}{s} + \frac{D}{s^2}$$

$$A(s-1)s^2 + B(s-9)s^2 + C(s-9)(s-1)s + D(s-9)(s-1)$$

$$= -s^3 + 12s + 5$$

for $s = 9$:

$$A(9-1)9^2 = -9^3 + 12 \cdot 9 + 5$$

$$648A = 248$$

$$A = \frac{248}{648} = \frac{31}{81}$$

for $s = 1$:

$$B(1-9)1^2 = -1^3 + 12 + 5$$

$$-8B = 12$$

$$B = -2$$

$s=0$ gives:

$D(-9)(-1)=5$

$D=\frac{5}{9}$

For C we simply plug the values of A, B and D into the earlier equation, giving:

$$-s^3+12s^2+5=\frac{31}{81}s^2(s-1)-2s^2(s-9)+Cs(s-9)(s-1)$$
$$+\frac{5}{9}(s-9)(s-1)$$
$$=Cs^3-\frac{131}{81}s^3-10Cs^2+\frac{1472}{81}s^2+9Cs$$
$$-\frac{50}{9}s+5$$
$$=s^3\left(C-\frac{131}{81}\right)+s^2\left(\frac{1472}{81}-10C\right)+s\left(9C-\frac{50}{9}\right)+5$$

since there is no s-term, we solve

$$9C-\frac{50}{9}=0$$
$$C=\frac{50}{9}\cdot\frac{1}{9}=\frac{50}{81}$$

so the end result is:

$$\mathcal{L}(y)=\frac{5}{9s^2}-\frac{2}{s-1}+\frac{50}{81s}+\frac{31}{81(s-9)}$$

Now taking the inverse Laplace transform of both sides we obtain

$$y=\mathcal{L}^{-1}\left(\frac{5}{9s^2}-\frac{2}{s-1}+\frac{50}{81s}+\frac{31}{81(s-9)}\right)$$
$$=\frac{5}{9}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)-2\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)+\frac{50}{81}\mathcal{L}^{-1}\left(\frac{1}{s}\right)+\frac{31}{81}\mathcal{L}^{-1}\left(\frac{1}{s-9}\right)$$
$$=\frac{5}{9}t-2e^{t}+\frac{50}{81}+\frac{31}{81}e^{9t}=y$$