Vraag 1

(a) $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^m}{\sqrt{m}}$

let op: quotient-test * werkt hier niet

alternerende reeks
met dalende coëfficiënten

$\Rightarrow$ convergent

(b) $\displaystyle\sum_{m=0}^{\infty} \frac{m^3}{(m+1)!}$

$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{(m+1)^3}{(m+2)!} \cdot \frac{(m+1)!}{m^3}$$

$$= \frac{(m+1)^3}{m^3} \cdot \frac{1}{m+2}$$

$$= \frac{m^3 + \ldots m^2 + \ldots m + \ldots}{m^3} \cdot \frac{1}{m+2}$$

$$= \underbrace{\left(1 + \ldots \frac{1}{m} + \ldots \frac{1}{m^2} + \ldots \frac{1}{m^3}\right)}_{\to 1} \cdot \underbrace{\frac{1}{m+2}}_{\to 0}$$

$m \to \infty$

$= 0 < 1 \Rightarrow$ convergent

(c) $\displaystyle\sum_{m=0}^{\infty} (m+1)^2$

divergent, want $\displaystyle\lim_{m\to\infty} (m+1)^2 \neq 0$

*(of via $\int$-test)

(d) $\displaystyle\sum_{m=1}^{\infty} \frac{1}{m^5}$

let op $\rightarrow$ $\left|\frac{a_{m+1}}{a_m}\right| = \ldots = \dfrac{1}{1 + \ldots \frac{1}{m} + \ldots + \ldots \frac{1}{m^5}} \to 1$

dus ratio test heeft hier geen zin!

wel zin heeft

$$\int_1^{\infty} \frac{dx}{x^5} = -\frac{1}{4} x^{-4} \Big|_1^{\infty} = \frac{1}{4} < \infty \Rightarrow \text{convergent}$$

of * via "comparison"-test

Vraag 2

$$T(t) = 1 - \cos(3t)$$
$$\mathcal{E}(t) = t \cdot \ln(1+2t)$$

(a)

$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \cdots$$

$$\Rightarrow 1 - \cos(3t) = \frac{(3t)^2}{2!} - \frac{(3t)^4}{4!} + \frac{(3t)^6}{6!} \cdots$$

$$= \underline{\frac{9}{2}t^2 - \frac{81}{24}t^4} + \frac{3^6}{6!}t^6 \cdots \leftarrow$$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} \cdots$$

$$\Rightarrow t \cdot \ln(1+2t) = t\left(2t - \frac{(2t)^2}{2} + \frac{(2t)^3}{3} - \frac{(2t)^4}{4} \cdots\right)$$

$$= \underline{2t^2 - 2t^3 + \frac{8}{3}t^4} - \frac{1}{4}t^5 \cdots \leftarrow$$

(b)

$$\lim_{t\to 0} \frac{T(t)}{\mathcal{E}(t)} = \lim_{t\to 0} \frac{\frac{9}{2}t^2 - \frac{81}{24}t^4 + \cdots}{2t^2 - 2t^3 + \frac{8}{3}t^4 \cdots}$$

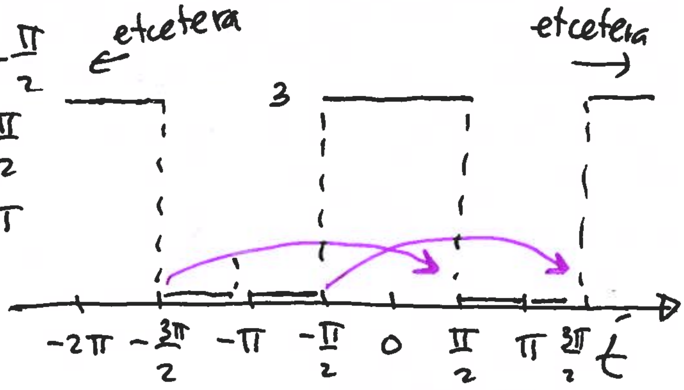
$$= \lim_{t\to 0} \frac{\frac{9}{2} - \frac{81}{24}t^2 \cdots}{2 - 2t + \cdots}$$

$$= \frac{9}{4}$$

Vraag 3

2π-periodiek

$$f(t) = \begin{cases} 0 & , -\pi < t < -\frac{\pi}{2} \\ 3 & , -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & , \frac{\pi}{2} < t < \pi \end{cases}$$



2π-periodiek

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$$

observatie: f is een <u>even</u> functie $\Rightarrow$ alle $b_n = 0$

dwz $b_0 = b_1 = b_2 = \dots = 0$

$$a_0 = \frac{2}{\pi} \int_0^{\pi/2} 3 \, dt = \frac{6}{\pi} \frac{\pi}{2} = 3$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} 3 . \cos(t) dt = \frac{2}{\pi} . 3 . \int_0^{\pi/2} \cos(t) dt$$

$$= \frac{6}{\pi} \sin(t) \Big|_0^{\pi/2} = \frac{6}{\pi}$$

$$a_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} 3 . \cos(2t) dt = \frac{2}{\pi} . 3 \frac{\sin(2t)}{2} \Big|_0^{\pi/2} = \frac{6}{\pi} . \frac{\left(\sin\left(\frac{2\pi}{2}\right) - \sin(0)\right)}{2}$$

$$= 0$$

Vraag 4

$$y^3 y' = x^5, \quad x > 0 \text{ en } y>0, \quad y(0) = 2$$

$$y^3 \frac{dy}{dx} = x^5$$

$$\Leftrightarrow \int y^3 \, dy = \int x^5 \, dx$$

$$\Leftrightarrow \frac{1}{4} y^4 = \frac{1}{6} x^6 + \tilde{c}$$

$$\Leftrightarrow y = \sqrt[4]{\frac{2}{3} x^6 + c} \qquad (y>0 \text{ gegeven})$$

dus "$-\sqrt{}$oplossing" telt niet mee hier

$$y(0) = \sqrt[4]{c} = 2 \Rightarrow c = 16$$

en dus $y(x) = \sqrt[4]{\frac{2}{3} x^6 + 16}$

Vraag 5

$$y'' + 3y' - 4y = \sin(2t) \quad ; \; y(0) = -\frac{3}{50} \; , \; y'(0) = \frac{21}{25}$$

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1) homogene oplossing:

$$y_h'' + 3y_h' - 4y_h = 0 \qquad \lambda^2 + 3\lambda - 4 = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -4 \end{cases}$$

$$\Rightarrow y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$= c_1 e^{t} + c_2 e^{-4t}$$

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particuliere oplossing:

2) $y_p = A\cos(2t) + B\sin(2t)$ (zowel cos als sin nodig !!)

$$y_p' = -2A\sin(2t) + 2B\cos(2t)$$
$$y_p'' = -4A\cos(2t) - 4B\sin(2t)$$

invullen

$$\Rightarrow -4A\cos(2t) - 4B\sin(2t) - 6A\sin(2t) + 6B\cos(2t) - 4A\cos(2t) - 4B\sin(2t) = \sin(2t)$$

$$\Rightarrow \begin{cases} -4A + 6B - 4A = 0 \\ -4B - 6A - 4B = 1 \end{cases} \Leftrightarrow \begin{cases} -8A + 6B = 0 \\ -8B - 6A = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{3}{50} \\ B = -\frac{2}{25} \end{cases}$$

$$\Rightarrow y_p(t) = -\frac{3}{50}\cos(2t) - \frac{2}{25}\sin(2t)$$

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volledige oplossing:

3) $y(t) = y_h(t) + y_p(t) = C_1 e^{t} + C_2 e^{-4t} - \frac{3}{50}\cos(2t) - \frac{2}{25}\sin(2t)$

$$y'(t) = C_1 e^{t} - 4C_2 e^{-4t} + \frac{3}{25}\sin(2t) - \frac{4}{25}\cos(2t)$$

$$\left.\begin{aligned} y(0) &= C_1 + C_2 - \frac{3}{50} - 0 = -\frac{3}{50} \Rightarrow C_1 = -C_2 \\ y'(0) &= C_1 - 4C_2 + 0 - \frac{4}{25} = \frac{21}{25} \Rightarrow C_1 - 4C_2 = 1 \end{aligned}\right\} \Rightarrow \begin{cases} C_1 = 1/5 \\ C_2 = -1/5 \end{cases}$$

$$\Rightarrow y(t) = \frac{1}{5}e^{t} - \frac{1}{5}e^{-4t} - \frac{3}{50}\cos(2t) - \frac{2}{25}\sin(2t)$$

Vraag 6

$$\begin{cases} x' = -y + xy \\ y' = x + \frac{1}{2}(x^2 - y^2) \end{cases}$$

(a) kritieke punten: $\begin{cases} y(-1+x) = 0 \\ x + \frac{1}{2}(x^2-y^2) = 0 \end{cases} \rightarrow x=1$ of $y=0$

$1 + \frac{1}{2}(1-y^2) = 0 \quad \downarrow \quad y = \pm\sqrt{3}$

$x + \frac{1}{2}x^2 = 0 \quad \downarrow \quad x=0,\ x=-2$

$$\Longleftarrow \begin{cases} (0,0) \\ (-2,0) \\ (1,\sqrt{3}) \\ (1,-\sqrt{3}) \end{cases}$$

(b) $J = \begin{pmatrix} y & -1+x \\ 1+x & -y \end{pmatrix}$

$(0,0)$: $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $sp = 0$, $\det > 0 \Rightarrow$ centrumpunt

$(-2,0)$: $J = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix}$, $\det < 0 \Rightarrow$ zadelpunt

$(1,\sqrt{3})$: $J = \begin{pmatrix} \sqrt{3} & 0 \\ 2 & -\sqrt{3} \end{pmatrix}$, $\det < 0$ " " "

$(1,-\sqrt{3})$: $J = \begin{pmatrix} -\sqrt{3} & 0 \\ 2 & \sqrt{3} \end{pmatrix}$, $\det < 0$ " " "

(c) lijn $y = \frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{3}}$ (*) voor alle (x,y) op die lijn

DV: stelsel $\frac{dy}{dx} = \frac{x + \frac{1}{2}(x^2 - y^2)}{(-1+x)y} = \frac{x + \frac{1}{2}\left(x^2 - \left(\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}\right)^2\right)}{(-1+x)\left(\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}\right)}$

$$= \frac{x + \frac{1}{2}x^2 - \frac{1}{2}\left(\frac{1}{3}x^2 + \frac{4}{3}x + \frac{4}{3}\right)}{-\frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}x^2 + \frac{2}{\sqrt{3}}x}$$

$$= \frac{\frac{1}{3}x^2 + \frac{1}{3}x - \frac{2}{3}}{\frac{1}{\sqrt{3}}x^2 + \frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}}} = \frac{\frac{1}{3}(x^2 + x - 2)}{\frac{1}{\sqrt{3}}(x^2 + x - 2)} = \frac{1/3}{1/\sqrt{3}} = \frac{1}{3}\sqrt{3} = \frac{1}{\sqrt{3}}$$

= (*)

[idem: $y = -\frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}}$ en $x = 1$ (hoeft niet)]

(d) figuur: zie volgende blz.

→ de drie lijnen uit (c)

→ de vier kritieke punten uit (a)

→ het karakter van deze vier punten (zie (b))

→ de pijlen!!

check "1 pijl" is voldoende: $x = 2, y = 0$ bijv.

$$\Rightarrow \begin{cases} x' = 0 \\ y' = 2 + \frac{1}{2}(4 - 0) > 0 \end{cases} \Rightarrow$$

! en daarmee liggen alle richtingen vast!

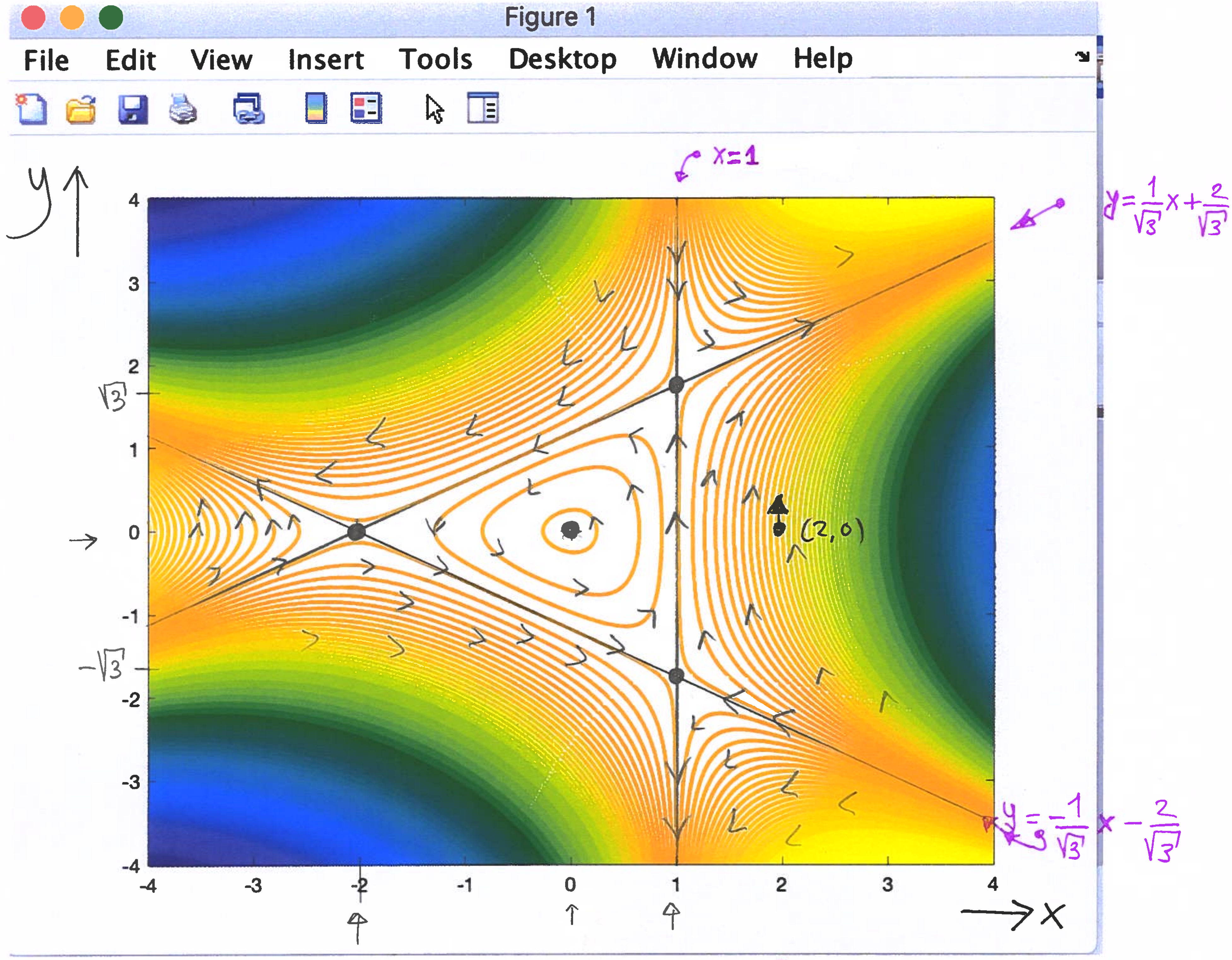

Figure 1
File Edit View Insert Tools Desktop Window Help
x=1
$y = \frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}$
$y = -\frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}}$
(2,0)
y
x
$\sqrt{3}$
$-\sqrt{3}$
4
3
2
1
0
-1
-2
-3
-4
-4 -3 -2 -1 0 1 2 3 4

Vraag 7

$$E = -\frac{1}{2}(x^2+y^2) + \frac{1}{2}\left(xy^2 - \frac{1}{3}x^3\right)$$

(a) $\nabla E = \text{grad}(E) = \begin{pmatrix} \frac{\partial E}{\partial x} \\ \frac{\partial E}{\partial y} \end{pmatrix} = \begin{pmatrix} -x + \frac{1}{2}y^2 - \frac{1}{2}x^2 \\ -y + xy \end{pmatrix}$

hoeft niet : (wel interessante observatie)

((Evt. vergelijk met vraag 6 !!) $\rightarrow \left[= \begin{pmatrix} -\left(x + \frac{1}{2}(x^2-y^2)\right) \\ -y + xy \end{pmatrix} \right]$))

$$\Delta E = \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = \frac{\partial}{\partial x}\left[-x + \frac{1}{2}y^2 - \frac{1}{2}x^2\right] + \frac{\partial}{\partial y}\left[-y+xy\right]$$

$= -1 - x - 1 + x = -2$ (voor alle x,y)

$$\mathcal{H} = \begin{pmatrix} \frac{\partial^2 E}{\partial x^2} & \frac{\partial^2 E}{\partial x \partial y} \\ \frac{\partial^2 E}{\partial y \partial x} & \frac{\partial^2 E}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -1-x & y \\ y & -1+x \end{pmatrix}$$

$\text{grad}(E)\Big|_{(2,1)} = \begin{pmatrix} -2+\frac{1}{2}-2 \\ -1+2 \end{pmatrix} = \begin{pmatrix} -7/2 \\ 1 \end{pmatrix}$

$$D_{\text{grad}(E)} E\Big|_{(2,1)} = \underbrace{\frac{\text{grad}(E)}{\|\text{grad}(E)\|}}_{\text{lengte}=1}\Bigg|_{(2,1)} \cdot \text{grad}(E)\Big|_{(2,1)} = \frac{2}{\sqrt{53}}\begin{pmatrix} -\frac{7}{2} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -\frac{7}{2} \\ 1 \end{pmatrix}$$

!!

$$= \frac{53}{4} \cdot \frac{2}{\sqrt{53}}$$

$$= \frac{53}{2\sqrt{53}} = \frac{1}{126}\sqrt{53}$$

$\|\text{grad}(E)\|_{\text{in }(2,1)}$

$= \sqrt{\left(-2+\frac{1}{2}-2\right)^2 + (-1+2)^2}$

$= \sqrt{\left(\frac{7}{2}\right)^2 + 1} = \sqrt{\frac{49}{4} + \frac{4}{4}} = \sqrt{\frac{53}{4}} = \frac{\sqrt{53}}{2}$

$\rightarrow$

(b) $\dfrac{\partial E}{\partial x}=0$ èn $\dfrac{\partial E}{\partial y}=0$

$$\Longleftrightarrow \begin{cases} -x+\frac{1}{2}y^2-\frac{1}{2}x^2=0 \\ -y+xy=0 \end{cases} \Longleftrightarrow \begin{cases} y(-1+x)=0 \\ x+\frac{1}{2}(x^2-y^2)=0 \end{cases}$$

hier

Vraag 6(a) !

dus dezelfde vier (x,y) punten als in Vraag 6(a) voldoen hier aan !

(c) $\det(\mathcal{H}) = \dfrac{\partial^2 E}{\partial x^2}\cdot\dfrac{\partial^2 E}{\partial y^2} - \left(\dfrac{\partial^2 E}{\partial x \partial y}\right)^2$

$$= (-1-x)(-1+x) - y^2$$

$$= 1 - x^2 - y^2$$

$\Delta E = -2$

zie (a)

stat. punt	ΔE	$\det(\mathcal{H})$	soort stat. punt
$(0,0)$	<0	>0	MAX
$(-2,0)$	<0	<0	ZADEL
$(1,\sqrt{3})$	<0	<0	ZADEL
$(1,-\sqrt{3})$	<0	<0	ZADEL

Vraag 8

$$y' = 2y + x^3 \quad , \; y(0) = 12$$

$$y = \sum_{n=0}^{\infty} c_n x^n \quad , \; y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n = 2\sum_{n=0}^{\infty} c_n x^n + x^3$$

uitschrijven:

$$c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots = 2c_0 + 2c_1 x + 2c_2 x^2 + 2c_3 x^3 + 2c_4 x^4 + \cdots + x^3$$

vegelijken coëfficiënten:

$$\begin{cases} c_1 = 2c_0 \\ 2c_2 = 2c_1 \\ 3c_3 = 2c_2 \\ 4c_4 = 2c_3 \underline{\underline{+1}} \\ 5c_5 = 2c_4 \\ \vdots \end{cases}$$

$y(0) = 12 = c_0$

$\Rightarrow c_1 = 24$

$c_2 = 24$

$c_3 = \frac{2}{3} . 24 = 16$

$c_4 = \frac{1}{2} . 16 + \frac{1}{4} = 8\frac{1}{4} = \frac{33}{4}$

$c_5 = \frac{2}{5} . \frac{33}{4} = \frac{33}{10}$

Vraag 9

$$u(x,y) = Ax^3 + xy^2 + Bx^2$$

$$\frac{\partial u}{\partial x} = 3Ax^2 + y^2 + 2Bx$$

$$\frac{\partial^2 u}{\partial x^2} = 6Ax + 2B$$

$$\frac{\partial u}{\partial y} = 2xy$$

$$\frac{\partial^2 u}{\partial y^2} = 2x$$

$$-\Delta u = -(6Ax + 2B + 2x) = x - 3$$

$$\Rightarrow \begin{cases} -6A - 2 = 1 \\ -2B = -3 \end{cases}$$

$$\Rightarrow A = -1/2, \quad B = 3/2$$

$$u(x,y) = -\tfrac{1}{2}x^3 + xy^2 + \tfrac{3}{2}x^2$$

rand_1 : $x=0 \Rightarrow u\big|_{x=0} = 0$

rand_2 : $x=1 \Rightarrow u\big|_{x=1} = -\tfrac{1}{2} + y^2 + \tfrac{3}{2} = y^2 + 1$

rand_3 : $y=0 \Rightarrow u\big|_{y=0} = -\tfrac{1}{2}x^3 + \tfrac{3}{2}x^2$

rand_4 : $y=1 \Rightarrow u\big|_{y=1} = -\tfrac{1}{2}x^3 + x + \tfrac{3}{2}x^2$