Section 7.2

Find (a) the general term and (b) the recurrence relation for the sequences:

1. 1, 4, 7, 10, ...

(a) $u_r = 1 + 3r;\quad r = 0, 1, 2, \ldots$ (b) $u_r = u_{r-1} + 3;\quad u_0 = 1$

2. 1, 3, 9, 27, ...

(a) $u_r = 3^r;\quad r = 0, 1, 2, \ldots$ (b) $u_r = 3u_{r-1};\quad u_0 = 1$

3. $1,\ -\frac{1}{5},\ \frac{1}{25},\ -\frac{1}{125},\ \ldots$

(a) $u_r = (-1/5)^r;\quad r = 0, 1, 2, \ldots$ (b) $u_r = (-1/5)u_{r-1};\quad u_0 = 1$

Find the first 6 terms of the sequences:

4. $u_{r+1} = u_r + \frac{1}{2};\quad u_1 = 0$

$$0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \frac{5}{2}$$

5. $v_n = \left(\frac{2}{3}\right)^n;\quad n = 0, 1, 2, \ldots$

$$1,\ \frac{2}{3},\ \frac{4}{9},\ \frac{8}{27},\ \frac{16}{81},\ \frac{32}{243}$$

6. $u_x = \frac{1}{x(x+2)};\quad x = 1, 2, 3, \ldots$

$$\frac{1}{1\cdot 3},\ \frac{1}{2\cdot 4},\ \frac{1}{3\cdot 5},\ \frac{1}{4\cdot 6},\ \frac{1}{5\cdot 7},\ \frac{1}{6\cdot 8}\ \rightarrow\ \frac{1}{3},\ \frac{1}{8},\ \frac{1}{15},\ \frac{1}{24},\ \frac{1}{35},\ \frac{1}{48}$$

7. $w_{n+1} = \frac{w_n}{n};\quad w_1 = 1$

$$1,\ \frac{1}{2},\ \frac{1}{6},\ \frac{1}{24},\ \frac{1}{120},\ \frac{1}{710}$$

8. $u_{n+2} = u_{n+1} + 2u_n;\quad u_0 = 1,\ u_1 = 3$

$$u_0 = 1,\ u_1 = 3,\ u_2 = u_1 + 2u_0 = 5,\ u_3 = u_2 + 2u_1 = 11,$$
$$u_4 = u_3 + 2u_2 = 21,\ u_5 = u_4 + 2u_3 = 43$$

9. $u_{n+2} = 3u_{n+1} - 2u_n$; $u_0 = 1$, $u_1 = 1/2$

$$u_0 = 1,\ u_1 = 1/2,\ u_2 = 3u_1 - 2u_0 = -1/2,\ u_3 = 3u_2 - 2u_1 = -5/2,$$
$$u_4 = 3u_3 - 2u_2 = -13/2,\ u_5 = 3u_4 - 2u_3 = -29/2$$

10. $u_{n+2} = 3u_{n+1} - 2u_n$; $u_0 = u_1$

$$u_0,\ u_1 = u_0,\ u_2 = 3u_0 - 2u_0 = u_0, \ldots$$
$$u_n = 3u_0 - 2u_0 = u_0, \text{ all } n$$

Find the limit $r \to \infty$ for:

11. $\dfrac{1}{3^r}$

$$\frac{1}{3^1} = \frac{1}{3},\ \frac{1}{3^2} = \frac{1}{9},\ \frac{1}{3^3} = \frac{1}{27},\ \frac{1}{3^4} = \frac{1}{81},\ \ldots \text{ and } \lim_{r\to\infty}\left(\frac{1}{3^r}\right) = 0$$

12. 2^r

$$2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ \ldots \text{ and } 2^r \to \infty \text{ as } r \to \infty$$

13. $\dfrac{1}{r+2}$

$$\frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \ldots \text{ and } \lim_{r\to\infty}\left(\frac{1}{r+2}\right) = 0$$

14. $\dfrac{r}{r+2}$

$$\frac{1}{3},\ \frac{2}{4},\ \frac{3}{5},\ \frac{4}{6},\ \ldots \text{ and } \frac{r}{r+2} \to \frac{r}{r} = 1 \text{ as } r \to \infty$$

15. $\dfrac{r}{r^2+r+1}$

Divide top and bottom by r: $\dfrac{r}{r^2+r+1} = \dfrac{1}{r+1+1/r} \to 0$ as $r \to \infty$

16. $\dfrac{3r^2+3r+1}{5r^2-6r-1}$

Divide top and bottom by r^2: $\dfrac{3r^2+3r+1}{5r^2-6r-1} = \dfrac{3+3/r+1/r^2}{5-6/r-1/r^2} \to \dfrac{3}{5}$ as $r \to \infty$

17. Find the limit of the sequence $\{u_{n+1}/u_n\}$ for $u_{n+2} = u_{n+1} + 2u_n$; $u_0 = 1$, $u_1 = 3$ (see Exercise 8).

We have $u_{n+2} = u_{n+1} + 2u_n \;\rightarrow\; \dfrac{u_{n+2}}{u_{n+1}} = 1 + 2\dfrac{u_n}{u_{n+1}}$

Let $\dfrac{u_{n+2}}{u_{n+1}} \rightarrow x$ as $n \rightarrow \infty$

Then $\dfrac{u_n}{u_{n+1}} = 1\Big/\left(\dfrac{u_{n+1}}{u_n}\right) \rightarrow \dfrac{1}{x}$ as $n \rightarrow \infty$

and $\dfrac{u_{n+2}}{u_{n+1}} = 1 + 2\dfrac{u_n}{u_{n+1}} \;\rightarrow\; x = 1 + \dfrac{2}{x}$

Solving for x,

$$x = 1 + \frac{2}{x} \;\rightarrow\; x^2 - x - 2 = (x-2)(x+1)$$
$$= 0 \text{ when } x = -1 \text{ or } x = 2$$

But the terms of the series are positive.

Therefore $x = \lim\limits_{n\rightarrow\infty} \dfrac{u_{n+1}}{u_n} = 2$

Section 7.3

Find the sum of **(i)** the first n terms, **(ii)** the first 10 terms:

18. $1 + 5 + 9 + 13 + \cdots$

(i) This is the arithmetic series with $a = 1$, $d = 4$, and sum $S_n = \dfrac{n}{2}\big[2 + 4(n-1)\big] = n(2n-1)$

(ii) $S_{10} = 10 \times 19 = 190$

19. $3 - 2 - 7 - 12 - \cdots$

(i) Arithmetic series with $a = 3$, $d = -5$, and sum $S_n = \dfrac{n}{2}\big[6 - 5(n-1)\big] = \dfrac{n}{2}\big[11 - 5n\big]$

(ii) $S_{10} = -5 \times 39 = -195$

20. $1 + 3 + 9 + 27 + \cdots$

(i) This is the geometric series with $a = 1$, $x = 3$, and sum $S_n = \dfrac{1-3^n}{1-3} = \dfrac{1}{2}(3^n - 1)$

(ii) $S_{10} = \dfrac{1}{2}(3^{10} - 1) = 29524$

21. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$

(i) Geometric series with $a=1$, $x=1/3$, and sum $S_n=\frac{1-(1/3)^n}{1-1/3}=\frac{3}{2}\left[1-(1/3)^n\right]$

(ii) $S_{10}==\frac{3}{2}\left[1-(1/3)^{10}\right]=1.5-\frac{1}{2\times 3^9}\approx 1.499975$

> Find the sum of the first n terms:

22. $x^3+x^5+x^7+\cdots=x^3\left[1+x^2+x^4+\cdots\right]$

Therefore $\quad S_n=x^3\left[\frac{1-x^{2n}}{1-x^2}\right]$

23. $x+2x^2+4x^3+\cdots=x\left[1+(2x)+(2x)^2+\cdots\right]$

Therefore $\quad S_n=x\left[\frac{1-(2x)^n}{1-2x}\right]$

> Use equation (7.11) to expand in powers of x:

24. $(1+x)^5=1+5x+\frac{5\times 4}{2}x^2+\frac{5\times 4\times 3}{3!}x^3+\frac{5\times 4\times 3\times 2}{4!}x^4+\frac{5\times 4\times 3\times 2\times 1}{5!}x^5$

$=1+5x+10x^2+10x^3+5x^4+x^5$

25. $(1+x)^7=1+7x+\frac{7\times 6}{2}x^2+\frac{7\times 6\times 5}{3!}x^3+\frac{7\times 6\times 5\times 4}{4!}x^4+\frac{7\times 6\times 5\times 4\times 3}{5!}x^5+\frac{7\times 6\times 5\times 4\times 3\times 2}{6!}x^6+\frac{7!}{7!}x^7$

$=1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7$

> Calculate the binomial coefficients $\binom{n}{r}$, $r=0,1,\ldots,n$, for

26. $n=3$

$$\binom{3}{0}=\frac{3!}{0!3!}=1,\quad \binom{3}{1}=\frac{3!}{1!2!}=3,\quad \binom{3}{2}=\frac{3!}{2!1!}=3,\quad \binom{3}{3}=\frac{3!}{3!0!}=1$$

34. **(i)** Calculate the distinct trinomial coefficients $\dfrac{4!}{n_1!n_2!n_3!}$.

(ii) Use the coefficients to expand $(a+b+c)^4$.

(i) The possible values of (n_1, n_2, n_3) for which $n_1 + n_2 + n_3 = 4$ are

(4, 0, 0), (0, 4, 0), (0, 0, 4),
(3, 1, 0), (3, 0, 1), (1, 3, 0), (1, 0, 3), (0, 3, 1), (0, 1, 3),
(2, 2, 0), (2, 0, 2), (0, 2, 2),
(2, 1, 1), (1, 2, 1), (1, 1, 2)

The corresponding distinct coefficients are

$$\frac{4!}{4!0!0!}=1,\quad \frac{4!}{3!1!0!}=4,\quad \frac{4!}{2!2!0!}=6,\quad \frac{4!}{2!1!1!}=12$$

(ii)
$$(a+b+c)^4 = \left(\frac{4!}{4!0!0!}\right)(a^4+b^4+b^4)+\left(\frac{4!}{3!1!0!}\right)(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)$$
$$+\left(\frac{4!}{2!2!0!}\right)(a^2b^2+a^2c^2+b^2c^2)+\left(\frac{4!}{2!1!1!}\right)(a^2bc+ab^2c+abc^2)$$
$$=(a^4+b^4+b^4)+4(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)$$
$$+6(a^2b^2+a^2c^2+b^2c^2)+12(a^2bc+ab^2c+abc^2)$$

35. **(i)** Calculate the distinct coefficients $\dfrac{3!}{n_1!n_2!n_3!n_4!}$.

(ii) Use the coefficients to expand $(a+b+c+d)^3$.

(i)
$$\frac{3!}{3!0!0!0!}=1,\quad \frac{3!}{2!1!0!0!}=3,\quad \frac{3!}{1!1!1!0!}=6$$

(ii)
$$(a+b+c+d)^3=(a^3+b^3+c^3+d^3)$$
$$+3(a^2b+a^2c+a^2d+b^2a+b^2c+b^2d+c^2a+c^2b+c^2d+d^2a+d^2b+d^2c)$$
$$+6(abc+abd+acd+bcd)$$

36. Find $\displaystyle\sum_{n=1}^{10}\frac{1}{n(n+1)}$.

By Example 7.6, $\displaystyle\sum_{n=1}^{N}\frac{1}{n(n+1)}=\frac{N}{N+1}$

$$=\frac{10}{11} \text{ when } N=10$$

39. **(i)** Verify that $(1+r)^3 - r^3 = 3r^2 + 3r + 1$, then

(ii) show that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

(i) $(1+r)^3 - r^3 = (r^3 + 3r^2 + 3r + 1) - r^3 = 3r^2 + 3r + 1$

(ii) We have

$$\sum_{r=1}^{n}(1+r)^3 - \sum_{r=1}^{n} r^3 = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 \qquad \text{(A)}$$

and

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \quad \sum_{r=1}^{n} 1 = n$$

On the left of the equal sign in (A),

$$\sum_{r=1}^{n}(1+r)^3 - \sum_{r=1}^{n} r^3 = \sum_{r=2}^{n+1} r^3 - \sum_{r=1}^{n} r^3$$

$$= \left[\sum_{r=1}^{n} r^3 - 1 + (n+1)^3\right] - \left[\sum_{r=1}^{n} r^3\right] = (n+1)^3 - 1$$

$$= n^3 + 3n^2 + 3n$$

Equation (A) is then

$$n^3 + 3n^2 + 3n = 3\sum_{r=1}^{n} r^2 + \frac{3}{2}n(n+1) + n$$

and

$$\sum_{r=1}^{n} r^2 = \frac{1}{3}\left[n^3 + 3n^2 + 3n - \frac{3}{2}n(n+1) - n\right] = \frac{1}{6}(2n^3 + 3n^2 + n)$$

$$= \frac{1}{6}n(2n+1)(n+1)$$

40. **(i)** Expand $(1+r)^6 - r^6$, then

(ii) use the series in Table 7.1 to find the sum of the series $\sum_{r=1}^{n} r^5$.

(i) $(1+r)^6 - r^6 = 1 + 6r + 15r^2 + 20r^3 + 15r^4 + 6r^5$

(ii) We have

$$\sum_{r=1}^{n}(1+r)^6 - \sum_{r=1}^{n} r^6 = \sum_{r=1}^{n} 1 + 6\sum_{r=1}^{n} r + 15\sum_{r=1}^{n} r^2 + 20\sum_{r=1}^{n} r^3 + 15\sum_{r=1}^{n} r^4 + 6\sum_{r=1}^{n} r^5 \qquad \text{(B)}$$

On the left of the equal sign in equation (B),

$$\sum_{r=1}^{n}(1+r)^6 - \sum_{r=1}^{n} r^6 = (1+n)^6 - 1 = 6n + 15n^2 + 20n^3 + 15n^4 + 6n^5 + n^6$$

On the right of the equal sign in equation (B), by Table 7.1,

$$\sum_{r=1}^{n} 1+6\sum_{r=1}^{n} r+15\sum_{r=1}^{n} r^2+20\sum_{r=1}^{n} r^3+15\sum_{r=1}^{n} r^4+6\sum_{r=1}^{n} r^5$$

$$= n+6\left[\frac{1}{2}n(n+1)\right]+15\left[\frac{1}{6}n(n+1)(2n+1)\right]+20\left[\frac{1}{4}n^2(n+1)^2\right]$$

$$+15\left[\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n\right]+6\sum_{r=1}^{n} r^5$$

$$= 6n+\frac{31}{2}n^2+20n^3+\frac{25}{2}n^4+3n^5+6\sum_{r=1}^{n} r^5$$

Hence $\displaystyle\sum_{r=1}^{n} r^5 = \frac{n^2}{12}\left[2n^4+6n^3+5n^2-1\right]$

Section 7.4

(i) Expand in powers of x to terms in x^6. **(ii)** Find the values of x for which the series converge:

41. (i) $\dfrac{1}{1-3x} = 1+(3x)+(3x)^2+(3x)^3+(3x)^4+(3x)^5+(3x)^6$ **(ii)** $|3x|<1 \to |x|<1/3$

$= 1+3x+9x^2+27x^3+81x^4+243x^5+729x^6$

42. (i) $\dfrac{1}{1+5x^2} = 1+(-5x^2)+(-5x^2)^2+(-5x^2)^3$ **(ii)** $|-5x^2|<1 \to |x|<1/\sqrt{5}$

$= 1-5x^2+25x^4-125x^6$

43. (i) $\dfrac{1}{2+x} = \dfrac{1}{2(1+x/2)} = \dfrac{1}{2}\left[1-\dfrac{x}{2}+\dfrac{x^2}{4}-\dfrac{x^3}{8}+\dfrac{x^4}{16}-\dfrac{x^5}{32}+\dfrac{x^6}{64}\right]$ **(ii)** $|x/2|<1 \to |x|<2$

44. (i) Use the geometric series to express the number $1/(10^6-1)$ as a decimal fraction. **(ii)** Show that the decimal representation of $1/7$ can be written as $142857/(10^6-1)$ (see Section 1.4)

(i) $$\frac{1}{10^6-1} = \frac{10^{-6}}{1-10^{-6}} = 10^{-6}\left[1+10^{-6}+10^{-12}+10^{-18}+\cdots\right]$$

$$= 10^{-6}+10^{-12}+10^{-18}+10^{-24}+\cdots$$

$= 0.000001 + 0.000000\,000001 + 0.000000\,000000\,000001 + \cdots$

$= 0.000001\,000001\,000001\ \ldots$

(ii) $$\frac{142857}{10^6-1} = 142857\times 0.000001\,000001\,000001\ \ldots$$

$$= 0.142857\,142857\,142857\ \ldots = \frac{1}{7}$$

45. The vibrational partition function of a harmonic oscillator is given by the series $q_{\text{v}} = \sum_{n=0}^{\infty} e^{-n\theta_{\text{v}}/T}$ where $\theta_{\text{v}} = h\nu_{\text{e}}/k$ is the vibrational temperature. Confirm that the series is a convergent geometric series, and find its sum.

We have $q_{\text{v}} = \sum_{n=0}^{\infty} e^{-n\theta_{\text{v}}/T} = \sum_{n=0}^{\infty} \left(e^{-\theta_{\text{v}}/T}\right)^n = \sum_{n=0}^{\infty} x^n$

Now $\theta_{\text{v}} > 0$ and $T > 0$, so that $\theta_{\text{v}}/T > 0$.

Therefore $x = e^{-\theta_{\text{v}}/T} < 1$

and the geometric series is convergent, with sum

$$q_{\text{v}} = (1-x)^{-1} = \left[1 - e^{-\theta_{\text{v}}/T}\right]^{-1}$$

Section 7.5

Examine the following series for convergence by

Comparison test

46. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

We have $\ln n < n$, $\frac{1}{\ln n} > n$, and each term of the series is greater than the corresponding term of the harmonic series. Therefore $\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$, and the series diverges.

47. $\sum_{r=1}^{\infty} \frac{\ln r}{r^3}$

We have $\ln r < r$, $\frac{\ln r}{r^3} < \frac{1}{r^2}$. Therefore $\sum_{r=1}^{\infty} \frac{\ln r}{r^3} < \sum_{r=1}^{\infty} \frac{1}{r^2}$, and the series converges.

D'Alembert ratio test

48. $\sum_{s=0}^{\infty} \frac{s^a}{(s+1)!}$

We have $u_s = \frac{s^a}{(s+1)!}$, $\frac{u_{s+1}}{u_s} = \left(\frac{s+1}{s}\right) \times \left(\frac{1}{s+2}\right) \to 0$ as $s \to \infty$, and the series converges for all values of a.

49. $\sum_{r=1}^{\infty} \frac{1}{r^a}$

We have $u_r = \frac{1}{r^a}$, $\frac{u_{r+1}}{u_r} = \left(\frac{r}{r+1}\right)^a \to 1$ as $r \to \infty$, and D'Alembert's ratio test fails (see Exercise 50).

Cauchy integral test:

50. $\sum_{r=1}^{\infty} \frac{1}{r^a}$

For $a = 1$, the harmonic series diverges.

For $a \neq 1$, $\int_1^{\infty} \frac{1}{r^a} dr = \lim_{b\to\infty} \left[-\frac{1}{(a-1)r^{a-1}} \right]_1^b \begin{cases} = \frac{1}{a-1} \text{ if } a > 1 \\ \text{divergees if } a < 1 \end{cases}$

The series therefore converges if $a > 1$, diverges if $a \le 1$.

51. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Let $x = \ln n$.

Then $\int_2^{\infty} \frac{1}{n \ln n} dn = \int_{\ln 2}^{\infty} \frac{1}{x} dx = \lim_{b\to\infty} \left[\ln x \right]_{\ln 2}^{b}$, and the series diverges.

Section 7.6

Find the radius of convergence of each of the following series:

52. $\sum_{m=0}^{\infty} \frac{x^m}{4^m}$

Let $c_m = \frac{1}{4^m}$, $c_{m+1} = \frac{1}{4^{m+1}}$.

Then $R = \frac{c_m}{c_{m+1}} = 4$, and the series converges when $|x| < 4$

53. $\sum_{r=0}^{\infty} (-1)^r x^{2r}$

Let $c_r = (-1)^r$, $c_{r+1} = (-1)^{r+1}$.

Then $R = \left| \frac{c_r}{c_{r+1}} \right| = 1$, and the series converges when $x^2 < 1$, $|x| < 1$