Section 20.9

NOTE: The arithmetic for the following exercises is tedious. You are advised to use a spreadsheet or write your own computer programs to perform the tasks.

34. Apply Euler's method to the initial value problem

$$y'(x) = -y(x), \qquad y(0) = 1$$

with step sizes **(i)** $h = 0.2$, **(ii)** $h = 0.1$, **(iii)** $h = 0.05$ to calculate approximate values of $y(x)$ for $x = 0.2, 0.4, 0.6, 0.8, 1.0$. Compare these with the values obtained from the exact solution $y = e^{-x}$.

We have $f(x_n, y_n) = y'(x_n)$ in equation (20.54). Therefore

$$y_{n+1} = y_n + hy'(x_n) = y_n - hy_n$$

The results of applying the Euler recursion relation are summarized in Table 16.

Table		**(i)** $h = 0.2$		**(ii)** $h = 0.1$		**(iii)** $h = 0.05$	
x	exp($-x$)	y	error	y	error	y	error
0.00	1.000000	1.000000	0.000000	1.000000	0.000000	1.000000	0.000000
0.05	0.951229					0.950000	−0.001229
0.10	0.904837			0.900000	−0.004837	0.902500	−0.002337
0.15	0.860708					0.857375	−0.003333
0.20	0.818731	0.800000	−0.018731	0.810000	−0.008731	0.814506	−0.004225
0.25	0.778801					0.773781	−0.005020
0.30	0.740818			0.729000	−0.011818	0.735092	−0.005726
0.35	0.704688					0.698337	−0.006351
0.40	0.670320	0.640000	−0.030320	0.656100	−0.014220	0.663420	−0.006900
0.45	0.637628					0.630249	−0.007379
0.50	0.606531			0.590490	−0.016041	0.598737	−0.007794
0.55	0.576950					0.568800	−0.008150
0.60	0.548812	0.512000	−0.036812	0.531441	−0.017371	0.540360	−0.008452
0.65	0.522046					0.513342	−0.008704
0.70	0.496585			0.478297	−0.034856	0.487675	−0.008910
0.75	0.472367					0.463291	−0.009075
0.80	0.449329	0.409600	−0.039729	0.430467	−0.018862	0.440127	−0.009202
0.85	0.427415					0.418120	−0.009295
0.90	0.406570			0.387420	−0.019149	0.397214	−0.009355
0.95	0.386741					0.377354	−0.009387
1.00	0.367879	**0.327680**	**−0.040199**	**0.348678**	**−0.019201**	**0.358486**	**−0.009394**

The values and errors of $y(1.0)$ are (i) 0.327680 and −0.040199 for $h = 0.2$, (ii) 0.348678 and −0.019201 for $h = 0.1$, (iii) 0.358486 and −0.009394 for $h = 0.05$, compared to the exact value $e^{-1} = 0.367879$.

For initial value problems in Exercises 35 to 37, **(i)** apply Euler's method with step size $h = 0.1$ to compute an approximate value of $y(1)$, **(ii)** confirm the given exact solution and compute the error:

35. $y' = 2 - 2y, \quad y(0) = 0; \quad y = 1 - e^{-2x}$

(i) The Euler recursion relation is

$$y_{n+1} = y_n + h(2 - 2y_n)$$

and the results are summarized in Table 17.

(ii) If $y = 1 - e^{-2x}$ then $y' = 2e^{-2x} = 2 - 2y$, as required.

Then: exact value: $y(1) = 1 - e^{-2} = 0.864665$

approximate: $y(1) \approx y_{10} = 0.892626$

error: $\varepsilon_{10} = y_{10} - y(1) = 0.027961$

36. $y' = \dfrac{y^2}{x+1}, \quad y(0) = 1; \quad y = \dfrac{1}{1 - \ln(x+1)}$

(i) The Euler recursion relation is

$$y_{n+1} = y_n + h\frac{y_n{}^2}{x_n + 1}$$

and the results are summarized in Table 18.

(ii) If $y = \dfrac{1}{1 - \ln(x+1)}$ then $y' = \dfrac{-1}{(1 - \ln(x+1))^2} \times \dfrac{-1}{x+1} = \dfrac{y^2}{x+1}$.

Then: exact value: $y(1) = 1/(1 - \ln 2) = 3.258891$

approximate: $y(1) \approx y_{10} = 2.845387$

error: $\varepsilon_{10} = y_{10} - y(1) = -0.4135041$

Table 17	x_n	y_n
$n = 0$	0.00	0.000000
1	0.10	0.200000
2	0.20	0.360000
3	0.30	0.488000
4	0.40	0.590400
5	0.50	0.672320
6	0.60	0.737856
7	0.70	0.790285
8	0.80	0.832228
9	0.90	0.865782
10	1.00	**0.892626**

Table 18	x_n	y_n
$n = 0$	0.00	1.000000
1	0.10	1.100000
2	0.20	1.210000
3	0.30	1.332008
4	0.40	1.468489
5	0.50	1.622522
6	0.60	1.798027
7	0.70	2.000083
8	0.80	2.235397
9	0.90	2.513008
10	1.00	**2.845387**

Opgave D1:

(1) $y(x) = \frac{1}{2}\left(e^{-3x} + e^{x}\right)$

(2) $y(x) = (2 + 11x)\, e^{-5x}$

(3) $y(x) = \frac{1}{2}\sin(2x)$.

Opgave D2:

$\varepsilon y'' - y = 0 \implies \varepsilon v' - v = 0 \implies v(x) = A e^{\frac{x}{\varepsilon}} \implies$

$y(x) = \int v(x)\, dx = A\varepsilon e^{\frac{x}{\varepsilon}} + C$

Gebruik randvoorwaarden om A en C te bepalen

Oplossing: $y(x) = \dfrac{e^{\frac{x}{\varepsilon}} - 1}{e^{\frac{1}{\varepsilon}} - 1}$

Opgave D4.

(a) $q(t_n + \Delta t) = q(t_n) + \Delta t\, \dot{q}(t_n) + \frac{(\Delta t)^2}{2}\ddot{q}(t_n) + \ldots$

Δt vervangen door $-\Delta t$ in bovenstaande Taylorreeks geeft een nieuwe Taylorreeks. Tel ze bij elkaar op en los de resulterende vergelijking op voor $\ddot{q}(t_n)$.

(b) Exact dezelfde procedure, maar trek de twee Taylorreeksen nu van elkaar af

Opgave D7:

Euler-forward:

$$\begin{cases} x_{n+1} = x_n + \Delta t\, y_n \\ y_{n+1} = y_n - \Delta t\, x_n \end{cases}$$

$$x_{n+1}^2 + y_{n+1}^2 = \left[1 + (\Delta t)^2\right](x_n^2 + y_n^2)$$

Euler-Backward:

$$\begin{cases} x_{n+1} = x_n + \Delta t\, y_{n+1} \\ y_{n+1} = y_n - \Delta t\, x_{n+1} \end{cases}$$

$$x_{n+1}^2 + y_{n+1}^2 = \frac{x_n^2 + y_n^2}{1 + (\Delta t)^2}$$

Verlet: (alleen formules)

$$\begin{cases} x_{n+1} = \left[2 - (\Delta t)^2\right] x_n - x_{n-1} \\ y_{n+1} = \left[2 - (\Delta t)^2\right] y_n - y_{n-1} \end{cases}$$

Impliciete midpuntmethode: (alleen energie)

$$x_{n+1}^2 + y_{n+1}^2 = x_n^2 + y_n^2$$