Section 13.2

Use the power-series method to solve:

1. $y' - 3x^2 y = 0$

By equation (13.2), we express the solution as the power series

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Then
$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$
$$= a_1 + 2a_2 x + \sum_{m=0}^{\infty} (m+3) a_{m+3} x^{m+2}$$

Also
$$3x^2 y = \sum_{m=0}^{\infty} 3a_m x^{m+2} = 3a_0 x^2 + 3a_1 x^3 + 3a_2 x^4 + \cdots$$

Therefore
$$y' - 3x^2 y = a_1 + 2a_2 x + \sum_{m=0}^{\infty} \left[(m+3)a_{m+3} - 3a_m\right] x^{m+2}$$

$= 0$ when the coefficient of each power of x is zero.

Then $a_1 = a_2 = 0$

and $(m+3)a_{m+3} - 3a_m = 0$ for $m = 0, 1, 2, \ldots$

$\rightarrow$ $3a_3 - 3a_0 = 0 \rightarrow a_3 = a_0$ (a_0 arbitrary)

$4a_4 - 3a_1 = 0 \rightarrow a_4 = 0$

$5a_5 - 3a_2 = 0 \rightarrow a_5 = 0$

$6a_6 - 3a_3 = 0 \rightarrow a_6 = \frac{1}{2}a_3 = \frac{1}{2!}a_0$

$7a_7 - 3a_4 = 0 \rightarrow a_7 = 0$

$8a_8 - 3a_5 = 0 \rightarrow a_8 = 0$

$9a_9 - 3a_6 = 0 \rightarrow a_9 = \frac{1}{3}a_6 = \frac{1}{3!}a_0$

and so on.

Therefore $a_{3n+1} = a_{3n+2} = 0$

$a_{3n} = \frac{1}{n!}a_0$ for $n = 0, 1, 2, \ldots$

and the power series expansion is

$$y = a_0\left[1 + x^3 + \frac{1}{2!}x^6 + \frac{1}{3!}x^9 + \cdots\right] = a_0\sum_{n=0}^{\infty}\frac{1}{n!}(x^3)^n$$

We recognise the sum as the power-series expansion of the function e^{x^3}. Therefore

$$y = a_0 e^{x^3}$$

where a_0 is an arbitrary constant.

2. $(1-x)y' - y = 0$. Confirm the solution can be expressed as $y = a/(1-x)$ when $|x| < 1$.

Let $$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Then $$(1-x)y' = \sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=1}^{\infty} m a_m x^m$$

$$= a_1 + (2a_2 - a_1)x + (3a_3 - 2a_2)x^2 + \cdots$$

$$= a_1 + \sum_{m=1}^{\infty}\left[(m+1)a_{m+1} - m a_m\right]x^m$$

Therefore $$(1-x)y' - y = a_1 - a_0 + \sum_{m=1}^{\infty}\left[(m+1)a_{m+1} - m a_m\right]x^m - \sum_{m=1}^{\infty} a_m x^m$$

$$= a_1 - a_0 + \sum_{m=1}^{\infty}(m+1)(a_{m+1} - a_m)x^m$$

$= 0$ when $a_{m+1} = a_m$ for all values of m.

All the coefficients are therefore equal, to arbitrary a_0 say, and the power series expansion is

$$y = a_0 \sum_{m=0}^{\infty} x^m$$

and this is recognized as the geometric series expansion of $\dfrac{a_0}{1-x}$, convergent when $|x| < 1$.

3. $y'' - 9y = 0$. Confirm that the solution can be expressed as $y = ae^{3x} + be^{-3x}$.

Let $$y = \sum_{m=0}^{\infty} a_m x^m$$

Then $$y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m$$

and $$y'' - 9y = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=0}^{\infty} 9a_m x^m$$

$$= 0 \text{ when } (m+2)(m+1)a_{m+2} = 9a_m$$

The recurrence relation for the coefficients gives rise to two independent series:

(i) m even $m = 0 \rightarrow 2a_2 = 9a_0 \rightarrow a_2 = \frac{9}{2}a_0$

$m = 2 \rightarrow 3 \times 4a_4 = 9a_2 \rightarrow a_4 = \frac{9^2}{4!}a_0$

$m = 4 \rightarrow 5 \times 6a_6 = 9a_4 \rightarrow a_6 = \frac{9^3}{6!}a_0$ and so on

Therefore, for even powers of x,

$$y_1(x) = a_0\left[1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \frac{(3x)^6}{6!} + \cdots\right]$$

(ii) m odd $m = 1 \rightarrow 2 \times 3a_3 = 9a_1 \rightarrow a_3 = \frac{9}{3!}a_1$

$m = 3 \rightarrow 4 \times 5a_5 = 9a_3 \rightarrow a_5 = \frac{9^2}{5!}a_1$

$m = 5 \rightarrow 6 \times 7a_7 = 9a_5 \rightarrow a_6 = \frac{9^3}{7!}a_1$ and so on

Therefore, for odd powers of x,

$$y_2(x) = \frac{a_1}{3}\left[(3x) + \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} + \frac{(3x)^7}{7!} + \cdots\right]$$

We recognize y_1 and y_2 as the hyperbolic functions

$$y_1(x) = a_0 \cosh 3x = \frac{a_0}{2}\left[e^{3x} + e^{-3x}\right], \quad y_2(x) = \frac{a_1}{3}\sinh 3x = \frac{a_1}{6}\left[e^{3x} - e^{-3x}\right]$$

Therefore $y(x) = y_1(x) + y_2(x) = a_0 \cosh 3x + (a_1/3)\sinh 3x$

$$= ae^{3x} + be^{-3x} \quad \text{where } a_0 = a + b,\ a_1/3 = a - b$$

4. $(1-x^2)y''-2xy'+2y=0$ (The Legendre equation (13.13) for $l=1$).

Show that the solution can be written as $y=a_1x+a_0\left[1+\dfrac{x}{2}\ln\left(\dfrac{1-x}{1+x}\right)\right]$ when $|x|<1$.

Let $\quad y=\sum_{m=0}^{\infty}a_m x^m,\quad y'=\sum_{m=1}^{\infty}ma_m x^{m-1},\quad y''=\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}$

Then $\quad (1-x^2)y''-2xy'+2y$

$$=\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}-\sum_{m=2}^{\infty}m(m-1)a_m x^m-2\sum_{m=1}^{\infty}ma_m x^m+2\sum_{m=0}^{\infty}a_m x^m$$

$$=\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m-\sum_{m=0}^{\infty}m(m-1)a_m x^m-2\sum_{m=0}^{\infty}ma_m x^m+2\sum_{m=0}^{\infty}a_m x^m$$

$$=\sum_{m=0}^{\infty}(m+2)\big[(m+1)a_{m+2}-(m-1)a_m\big]x^m$$

and $\quad (1-x^2)y''-2xy'+2y=0\quad$ when $\quad a_{m+2}=\left(\dfrac{m-1}{m+1}\right)a_m$

The recurrence relation for the coefficients gives rise to two independent series:

(i) m even: $a_2=-a_0,\ a_4=\dfrac{a_2}{3}=-\dfrac{a_0}{3},\ a_6=\dfrac{3a_4}{5}=-\dfrac{a_0}{5},\ \cdots$ with a_0 arbitrary

$\rightarrow\ a_n=-\dfrac{a_0}{n-1}$ for $n=2,4,6,\ldots$

(ii) m odd: a_1 arbitrary,

$a_3=0,\ a_5=0\ \rightarrow\ a_n=0$ for odd $n>1$

Therefore $\quad y(x)=a_1x+a_0\left[1-x^2-\dfrac{x^4}{3}-\dfrac{x^6}{5}-\dfrac{x^8}{7}-\cdots\right]$

Now

$$1+\frac{x}{2}\ln\left(\frac{1-x}{1+x}\right)=1+\frac{x}{2}\big[\ln(1-x)=\ln(1+x)\big]$$

$$=1+\frac{x}{2}\left[\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\cdots\right)-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots\right)\right]$$

$$=1+\frac{x}{2}\left[-2x-\frac{2x^3}{3}-\frac{2x^5}{5}\cdots\right]=1-x^2-\frac{x^4}{3}-\frac{x^6}{5}-\cdots\quad\text{when } |x|<1$$

Therefore $\quad y=a_1x+a_0\left[1+\dfrac{x}{2}\ln\left(\dfrac{1-x}{1+x}\right)\right]$ when $|x|<1$

5. $y'' - xy = 0$ (Airy equation).

Let $$y = \sum_{m=0}^{\infty} a_m x^m \quad \rightarrow \quad xy = \sum_{m=0}^{\infty} a_m x^{m+1}$$

Then $$y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + \sum_{m=0}^{\infty} (m+3)(m+2)a_{m+3} x^{m+1}$$

and $$y'' - xy = 2a_2 + \sum_{m=0}^{\infty} (m+3)(m+2)a_{m+3} x^{m+1} - \sum_{m=0}^{\infty} a_m x^{m+1}$$

$$= 2a_2 + \sum_{m=0}^{\infty} \left[(m+3)(m+2)a_{m+3} - a_m \right] x^{m+1}$$

$$= 0 \text{ when } a_2 = 0 \text{ and } a_{m+3} = \frac{1}{(m+3)(m+2)} a_m$$

The recurrence relation for the coefficients gives rise to three independent series:

$$a_3 = \frac{a_0}{2\times 3} = \frac{1}{3!}a_0, \quad a_6 = \frac{a_3}{5\times 6} = \frac{1\times 4}{6!}a_0, \quad a_9 = \frac{a_6}{8\times 9} = \frac{1\times 4\times 7}{9!}a_0, \cdots$$

$$a_4 = \frac{a_1}{3\times 4} = \frac{2}{4!}a_1, \quad a_7 = \frac{a_4}{6\times 7} = \frac{2\times 5}{7!}a_1, \quad a_{10} = \frac{a_6}{9\times 10} = \frac{2\times 5\times 8}{9!}a_1, \cdots$$

$$a_5 = \frac{a_2}{4\times 5} = 0 = a_8 = a_{11} = \cdots$$

Therefore
$$y(x) = a_0\left[1 + \frac{1}{3!}x^3 + \frac{1\cdot 4}{6!}x^6 + \frac{1\cdot 4\cdot 7}{9!}x^9 + \cdots\right] + a_1\left[x + \frac{2}{4!}x^4 + \frac{2\cdot 5}{7!}x^7 + \frac{2\cdot 5\cdot 8}{10!}x^{10} + \cdots\right]$$

Section 13.3

For each of the following, find and solve the indicial equation

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6. $x^2y'' + 3xy' + y = 0$

We have $b_0 = 3,\ c_0 = 1$

and the indicial equation is

$$r^2 + (b_0 - 1)r + c_0 = 0 \quad \rightarrow \quad r^2 + 2r + 1 = (r+1)^2 = 0 \text{ when } r = -1$$

Therefore $r_1 = r_2 = -1$ (double root) optioneel

7. $x^2y'' + xy' + (x^2 - n^2)y = 0$ (Bessel equation)

We have $b_0 = 1,\ c_0 = -n^2$

and the indicial equation is

$$r^2 + (b_0 - 1)r + c_0 = 0 \ \rightarrow\ r^2 - n^2 = (r-n)(r+n) = 0 \text{ when } r = \pm n$$

8. $xy'' + (1-2x)y' + (x-1)y = 0$

We write the equation as $x^2y'' + (x - 2x^2)y' + (x^2 - x)y = 0$

Therefore $b_0 = 1,\ c_0 = 0 \ \rightarrow\ r^2 = 0$

Therefore $r_1 = r_2 = 0$

9. $x^2y'' + 6xy' + (6 - x^2)y = 0$

We have $b_0 = 6,\ c_0 = 6 \ \rightarrow\ r^2 + 5r + 6 = (r+2)(r+3) = 0$

Therefore $r_1 = -2,\ r_2 = -3$

10. **(i)** Find the general solution of the Euler-Cauchy equation $x^2y'' + b_0xy' + c_0y = 0$ for distinct indicial roots, $r_1 \neq r_2$. **(ii)** Show that for a double initial root *r*, the general solution is $y = (a + b\ln x)x^r$.

Let $$y(x) = x^r(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) = x^r\sum_{m=0}^{\infty} a_m x^m$$

Then, by equation (13.7)

$$\sum_{m=0}^{\infty}\Big[(r+m)^2 + (b_0 - 1)(r+m) + c_0\Big]a_m x^{m+r} = 0$$

The equation is satisfied if, for every power of *x*, either $a_m = 0$ or the term in square brackets is zero. For $m = 0$, the latter is the indicial equation (13.8), so that a particular solution of the differential equation is

$y = x^r$, where *r* is an indicial root. There are two possible types of solution.

(i) Distinct indicial roots, $r_1 \neq r_2$. Then by equations (13.10), x^{r_1} and x^{r_2} are two independent particular solutions, and the general solution of the Euler-Cauchy equation is

$$y(x) = ax^{r_1} + bx^{r_2}$$