Section 14.2

1. Show that the function $f(x,t) = a\sin(bx)\cos(vbt)$ **(i)** satisfies the 1-dimensional wave equation (14.1), **(ii)** has the form $f(x,t) = F(x+vt) + G(x-vt)$.

(i) We have $f(x,t) = a\sin(bx)\cos(vbt)$

Then
$$\frac{\partial f}{\partial x} = ba\cos(bx)\cos(vbt), \quad \frac{\partial^2 f}{\partial x^2} = -b^2 a\sin(bx)\cos(vbt) = -b^2 f$$
$$\frac{\partial f}{\partial t} = vba\sin(bx)\sin(vbt), \quad \frac{\partial^2 f}{\partial t^2} = -v^2 b^2 a\sin(bx)\sin(vbt) = -v^2 b^2 f$$

Therefore
$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

(ii) We have
$$\sin A\cos B = \frac{1}{2}\big[\sin(A+B) + \sin(A-B)\big]$$

Therefore
$$f(x,t) = a\sin(bx)\cos(vbt) = \frac{a}{2}\big[\sin(bx+vbt) + \sin(bx-vbt)\big]$$

2. The diffusion equation $\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$ provides a model of, for example, the transfer of heat from a hot region of a system to a cold region by conduction when $f(x,t)$ is a temperature field, or the transfer of matter from a region of high concentration to one of low concentration when f is the concentration. Find the functions $V(x)$ for which $f(x,t) = V(x)e^{ct}$ is a solution of the equation.

We have
$$f(x,t) = V(x)e^{ct}, \quad \frac{\partial f}{\partial t} = cV(x)e^{ct}, \quad \frac{\partial^2 f}{\partial x^2} = \frac{d^2V}{dx^2}e^{ct}$$

Then
$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \;\to\; cV(x)e^{ct} = D\frac{d^2V}{dx^2}e^{ct}$$
$$\to\; \frac{d^2V}{dx^2} = \frac{c}{D}V$$

The type solution depends on the value of c/D:

(a) $c/D = 0$ $\quad \frac{d^2V}{dx^2} = 0 \;\to\; V = a + bx$

(b) $c/D = \lambda^2 > 0$ $\quad c/D = \lambda > 0 \;\to\; V = ae^{\lambda x} + be^{-\lambda x}$

(c) $c/D = \lambda^2 > 0$ $\quad V = a\cos\lambda x + b\sin\lambda x$

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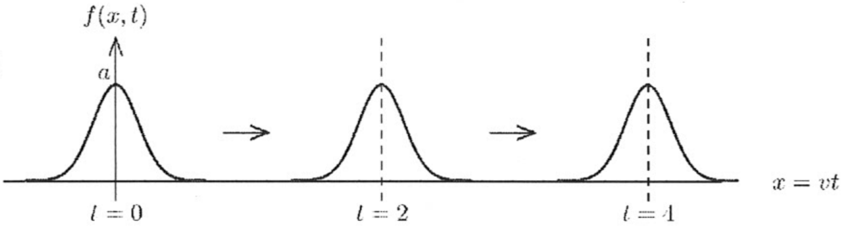
3. **(i)** It is shown in Example 14.2 that the function $f(x,t) = a\exp[-b(x-vt)^2]$ is a solution of the wave equation (14.1). Sketch graphs of $f(x,t)$ as a function of x at times $t=0$, $t=2/v$, $t=4/v$ (use, for example, $a=b=1$) to demonstrate that the function represents a wave travelling to the right (in the positive x–direction) at constant speed v.
(ii) Verify that $g(x,t) = a\exp[-b(x+vt)^2]$ is also a solution of the wave equation, and hence that every superposition $F(x,t) = f(x,t)+g(x,t)$ is a solution. **(iii)** Sketch appropriate graphs of $f(x,t)+g(x,t)$ to demonstrate how this function develops in time.

(i) The function $f(x,t) = a\exp[-b(x-vt)^2]$ represents a Gaussian wave whose centre lies at $x = vt$. The centre moves to the right (the positive x-direction) with constant speed $dx/dt = v$. Figure 1 shows the wave at times $t=0$, $t=2/v$, $t=4/v$.

Figure 1



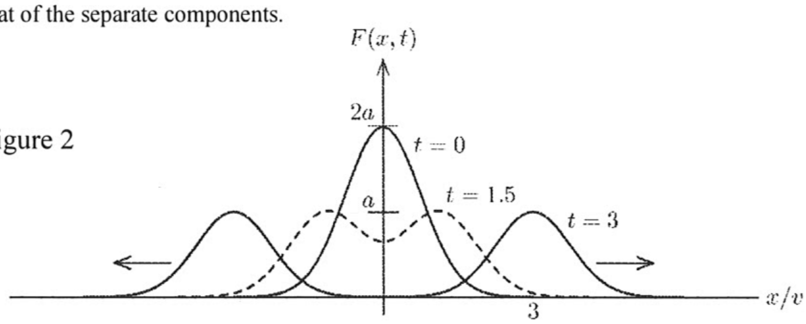
(ii) Function $g(x,t)$ is obtained from $f(x,t)$ by replacement if v by $-v$, and has the same second derivative with respect to time t, proportional to v^2, as in Example 14.2. Thus

$$\frac{\partial^2 g}{\partial x^2} = \frac{1}{(-v)^2}\frac{\partial^2 g}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2}$$

Then
$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2}{\partial x^2}(af+bg) = a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 g}{\partial x^2} = \frac{a}{v^2}\frac{\partial^2 f}{\partial t^2} + \frac{b}{v^2}\frac{\partial^2 g}{\partial t^2}$$
$$= \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(af+bg) = \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2}$$

(iii) In Figure 2, the component *f* of $F = f+g$ moves to the right with constant speed v, the component *g* to the left with the same speed; that is, the components separate as *t* increases. The amplitude of the total wave at $t=0$ is twice that of the components, but decrease with *t* to that of the separate components.

Figure 2



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Section 14.3

Find solutions of the following equations by the method of separation of variables:

4. $2\frac{\partial f}{\partial x}+\frac{\partial f}{\partial t}=0$

Let $f(x,t)=F(x)\times G(t)$

Then $\frac{\partial f}{\partial x}=\frac{dF(x)}{dx}\times G(t),\quad \frac{\partial f}{\partial t}=F(x)\times\frac{dG(t)}{dt}$

and $2\frac{\partial f}{\partial x}+\frac{\partial f}{\partial t}=0 \;\rightarrow\; 2\frac{dF(x)}{dx}G(t)+F(x)\frac{dG(t)}{dt}=0$

Division throughout by $f=F(x)\times G(y)$ gives

$$\left[\frac{2}{F(x)}\frac{dF(x)}{dx}\right]+\left[\frac{1}{G(t)}\frac{dG(t)}{dt}\right]=0$$

The two sets of terms in square brackets must be separately constant if x and t are independent variables. Therefore, if the first set of terms equals the constant C then the second set is equal to $-C$ (for the total to be zero). The resulting ordinary first-order equation in variable x is

$$\left[\frac{2}{F(x)}\frac{dF(x)}{dx}\right]=C \;\rightarrow\; \frac{dF(x)}{dx}=\frac{C}{2}F(x)$$

with general solution $F(x)=ae^{Cx/2}$. The corresponding equation in variable t is

$$\left[\frac{1}{G(t)}\frac{dG(t)}{dt}\right]=-C \;\rightarrow\; \frac{dG(t)}{dt}=-CG(t)$$

with general solution $G(t)=be^{-Ct}$. A complete solution is therefore

$$\begin{aligned}f(x,t)&=F(x)\times G(t)\\&=ae^{Cx/2}\times be^{-Ct}=abe^{C(x/2-t)}\\&=Ae^{B(x-2t)}\end{aligned}$$

where A and B are arbitrary constants.

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5. $y\dfrac{\partial f}{\partial x} - x\dfrac{\partial f}{\partial y} = 0$

Let $\quad f(x,y) = F(x)\times G(y) \;\to\; \dfrac{\partial f}{\partial x} = \dfrac{dF}{dx}\times G,\;\; \dfrac{\partial f}{\partial y} = F\times\dfrac{dG}{dy}$

Then $\quad y\dfrac{\partial f}{\partial x} - x\dfrac{\partial f}{\partial t} = 0 \;\to\; y\dfrac{dF}{dx}G - xF\dfrac{dG}{dy} = 0$

$$\to \left[\frac{1}{xF}\frac{dF}{dx}\right] - \left[\frac{1}{yG}\frac{dG}{dy}\right] = 0$$

Putting each set of terms equal to constant C, we have (see Section 11.3)

$$\frac{dF}{dx} = CxF \;\to\; \int\frac{dF}{F} = C\int x\,dx \;\to\; \ln F = C\frac{x^2}{2} + c$$

$$\to\; F = ae^{Cx^2/2}$$

Similarly, $\quad \dfrac{dg}{dy} = CyG \;\to\; G = be^{Cy^2/2}$

Therefore $\quad f(x,y) = abe^{C(x^2+y^2)/2} = Ae^{B(x^2+y^2)}$

6. $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$

Let $\quad f(x,y) = F(x)\times G(y) \;\to\; \dfrac{\partial^2 f}{\partial x^2} = \dfrac{d^2F}{dx^2}\times G,\;\; \dfrac{\partial^2 f}{\partial y^2} = F\times\dfrac{d^2G}{dy^2}$

Then $\quad \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0 \;\to\; \dfrac{d^2F}{dx^2}\times G + F\times\dfrac{d^2G}{dy^2} = 0 \;\to\; \left[\dfrac{1}{F}\dfrac{d^2F}{dx^2}\right] + \left[\dfrac{1}{G}\dfrac{d^2G}{dy^2}\right] = 0$

and $\quad \dfrac{d^2F}{dx^2} = CF, \qquad \dfrac{d^2G}{dy^2} = -CG$

As in Exercise 2, there are three possible types of solutions:

(a) $C = 0$: $\begin{cases} F(x) = a + bx \\ G(y) = c + dy \end{cases} \;\to\; f(x,y) = (a+bx)(c+dy)$

(b) $C = \lambda^2 > 0$: $\begin{cases} F(x) = ae^{\lambda x} + be^{-\lambda x} \\ G(y) = c\cos\lambda y + d\sin\lambda y \end{cases} \;\to\; f(x,y) = (ae^{\lambda x} + be^{-\lambda x})(c\cos\lambda y + d\sin\lambda y)$

(c) $C = \lambda^2 < 0$: $\begin{cases} F(x) = a\cos\lambda x + b\sin\lambda x \\ G(y) = ce^{\lambda y} + de^{-\lambda y} \end{cases} \;\to\; f(x,y) = (a\cos\lambda x + b\sin\lambda x)(ce^{\lambda y} + de^{-\lambda y})$

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7. $\dfrac{\partial^2 f}{\partial x \partial y} + f = 0$

We have $f(x, y) = F(x) \times G(y) \;\rightarrow\; \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{dF}{dx} \times \dfrac{dG}{dy}$

Then $\dfrac{\partial^2 f}{\partial x \partial y} + f = \dfrac{dF}{dx} \times \dfrac{dG}{dy} + FG = 0$ when $\left[\dfrac{1}{F}\dfrac{dF}{dx}\right]\left[\dfrac{1}{G}\dfrac{dG}{dy}\right] + 1 = 0$

and $\dfrac{dF}{dx} = CF \;\rightarrow\; F = ae^{Cx}, \qquad \dfrac{dG}{dy} = -\dfrac{1}{C}G \;\rightarrow\; G = be^{-y/C}$

Therefore $f(x, y) = Ae^{(Cx - y/C)}$

Section 14.4

8. Show that the wave functions (14.23) satisfy the orthonormality conditions

$$\int_0^b \int_0^a \psi_{p,q}(x, y)\psi_{r,s}(x, y)\,dx\,dy = \begin{cases} 1 & \text{if } p = r \text{ and } q = s \\ 0 & \text{otherwise} \end{cases}$$

We have $\psi_{p,q} = \sqrt{\dfrac{2}{a}}\sin\left(\dfrac{p\pi x}{a}\right) \times \sqrt{\dfrac{2}{b}}\sin\left(\dfrac{q\pi y}{b}\right), \quad \psi_{r,s} = \sqrt{\dfrac{2}{a}}\sin\left(\dfrac{r\pi x}{a}\right) \times \sqrt{\dfrac{2}{b}}\sin\left(\dfrac{s\pi y}{b}\right)$

Then
$$I = \int_0^b \int_0^a \psi_{p,q}(x, y)\psi_{r,s}(x, y)\,dx\,dy$$
$$= \frac{2}{a}\int_0^a \sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{r\pi x}{a}\right)dx \times \frac{2}{b}\int_0^b \sin\left(\frac{q\pi y}{b}\right)\sin\left(\frac{s\pi y}{b}\right)dy = I_{p,r} \times I_{q,s}$$

Remember $\sin Ax \sin Bx = \dfrac{1}{2}\left[\cos(A - B)x - \cos(A + B)\right]x$

Then, if $A = p\pi/a,\; B = r\pi/a$, where p and r are integers,

$$I_{p,r} = \frac{2}{a}\int_0^a \sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{r\pi x}{a}\right)dx = \begin{cases} \left[\dfrac{\sin(p-q)\pi}{(p-q)\pi} - \dfrac{\sin(p+q)\pi}{(p+q)\pi}\right] = 0 & \text{if } p \neq r \\ \dfrac{2}{a}\displaystyle\int_0^a \sin^2\left(\frac{p\pi x}{a}\right)dx = \left[1 - \dfrac{\sin 2p\pi}{2p\pi}\right] = 1 & \text{if } p = r \end{cases}$$

and similarly for $I_{q,s}$.

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