Section 15.4

7. Confirm the relations **(i)** (15.33), **(ii)** (15.34) and **(iii)** (15.36).

(i) $\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx = 0, \quad m \neq n$

We have $\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$

Therefore $\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx = \frac{1}{2}\int_{-\pi}^{+\pi} \left[\cos(m-n)x - \cos(m+n)x\right] dx$

$$= \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{+\pi}$$ (integral 4 in Table 6.1)

But $\sin px = 0$ when p is an integer.

Therefore $\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx = 0$

(ii) $\int_{-\pi}^{+\pi} \cos mx \sin nx \, dx = 0, \quad \text{all } m, n$

$\cos mx$ is an even function of x, $\sin nx$ is an odd function of x. The product $\cos mx \sin nx$ is therefore odd, and the integral is zero.

(iii) $\int_{-\pi}^{+\pi} \sin nx \sin nx \, dx = \pi \quad \text{if } n > 0$

We have $\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

Therefore $\int_{-\pi}^{+\pi} \sin^2 nx \, dx = \frac{1}{2}\int_{-\pi}^{+\pi} (1 - \cos 2nx) \, dx$

$$= \left[x - \frac{1}{2n}\sin 2nx\right]_{-\pi}^{+\pi} = \pi$$ (integral 2 in Table 6.1)

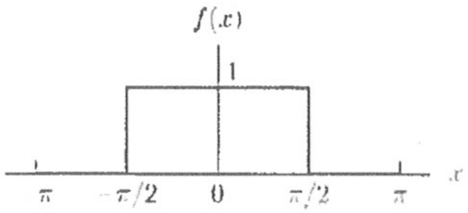
8. A periodic function with period 2π is defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{if} \quad \frac{\pi}{2} < |x| < \pi \end{cases}$$

(i) Draw the graph of the function in the interval $-3\pi \le x \le 3\pi$. **(ii)** Find the Fourier series of the function [Hint: $f(x)$ is an even function of x]. **(iii)** Use the series to show that

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ [Hint: substitute a suitable value for x in the series].

(i) Figure



(ii) The Fourier series (15.37) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

In the present case, $f(x)$ is an even function in the interval $-\pi < x < \pi$ and only the even trigonometric functions $\cos nx$ contribute (all $b_n = 0$):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{(Fourier cosine series)}$$

We have $\quad a_0 = \frac{2}{\pi} \int_0^{\pi/2} dx = 1$

and $\quad a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} \cos nx \, dx$

$$= \frac{2}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi/2} = \begin{cases} 0 & \text{if } n \text{ even} \\ 2/n\pi & \text{if } n = 1, 5, 9 \ldots \\ -2/n\pi & \text{if } n = 3, 7, 11 \ldots \end{cases}$$

Then $\quad f(x) = \frac{1}{2} + \frac{2}{\pi} \left[\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \cdots \right]$

(iii) Put $x = 0$

Then $\quad f(0) = 1 = \frac{1}{2} + \frac{2}{\pi} \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right]$

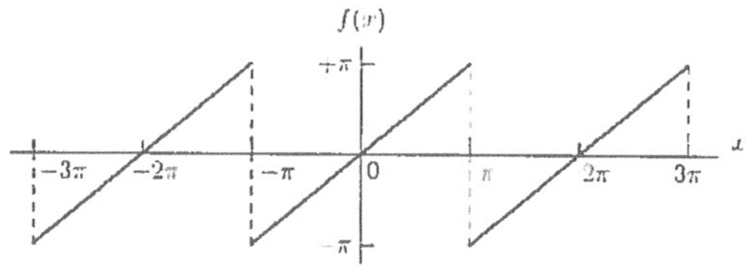
and $\quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

9. A function with period 2π is defined by

$$f(x) = x, \qquad -\pi < x < \pi$$

(i) Draw the graph of the function in the interval $-3\pi \le x \le 3\pi$. (ii) Find the Fourier series of the function. [Hint: $f(x)$ is an odd function of x] (iii) Draw the graphs of the first four partial sums of the series.

(i) Figure



(ii) Function $f(x)$ is an odd function in the interval $-\pi < x < \pi$ and only the odd trigonometric functions $\sin nx$ contribute to the Fourier series (all $a_n = 0$):

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{(Fourier sine series)}$$

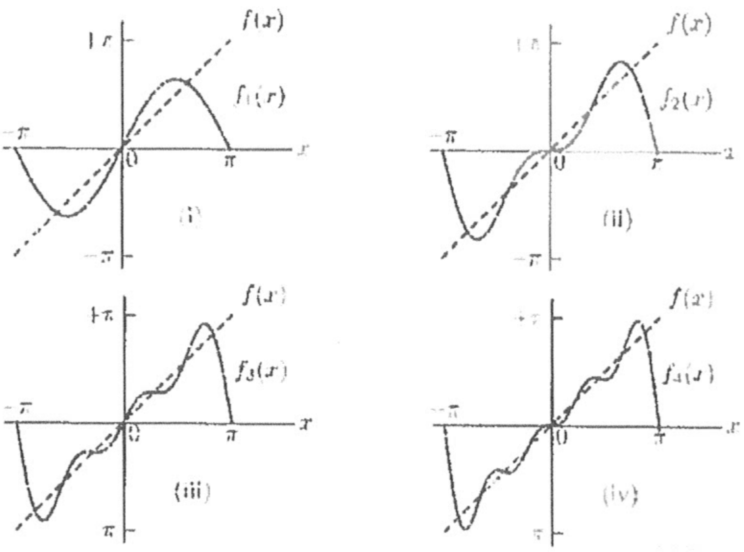
We have $$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx$$

$$= \frac{2}{\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{+\pi}\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_0^{\pi} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right\} = -\frac{2}{n}\cos n\pi \begin{cases} +2/n & \text{if } n \text{ odd} \\ -2/n & \text{if } n \text{ even} \end{cases}$$

Then $$f(x) = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

(iii) Figure



10. A function with period 2π is defined by

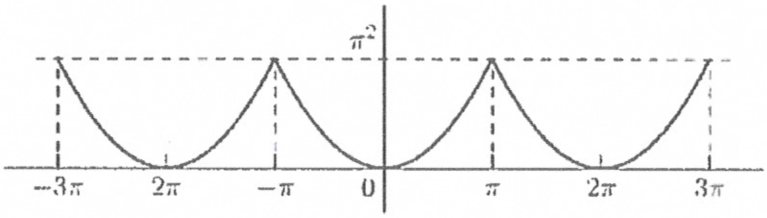
$$f(x) = x^2, \qquad -\pi \le x \le \pi$$

(i) Draw the graph of the function in the interval $-3\pi \le x \le 3\pi$. **(ii)** Find the Fourier series of the function. **(iii)** Use the series to show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

(i) Figure



(ii) Function $f(x)$ is an even function in the interval $-\pi < x < \pi$ and only the even trigonometric functions $\cos nx$ contribute:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Then $$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

and, by parts (as in Example 6.11),

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx = \frac{4}{n^2}\cos n\pi = \begin{cases} -4/n^2 & \text{if } n \text{ odd} \\ +4/n^2 & \text{if } n \text{ even} \end{cases}$$

Therefore $$f(x) = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right]$$

(iii) Put $x = \pi$: $$f(\pi) = \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Put $x = 0$: $$f(0) = 0 = \frac{\pi^2}{3} - 4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\rightarrow \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$