Section 12.2

1. Show that e^{-2x} and $e^{2x/3}$ are particular solutions of the differential equation $3y''+4y'-4y=0$.

We have $y=e^{-2x} \rightarrow y'=\dfrac{dy}{dx}=-2e^{-2x}=-2y \rightarrow y''=\dfrac{d^2y}{dx^2}=4e^{-2x}=4y$

Therefore $3y''+4y'-4y=\left[12-8-4\right]y=0$

Similarly $y=e^{2x/3} \rightarrow y'=\dfrac{2}{3}e^{2x/3}=\dfrac{2}{3}y \rightarrow y''=\dfrac{4}{9}e^{2x/3}=\dfrac{4}{9}y$

and $3y''+4y'-4y=\left[\dfrac{4}{3}+\dfrac{8}{3}-4\right]y=0$

2. Show that e^{3x} and xe^{3x} are particular solutions of the differential equation $y''-6y'+9y=0$.

$y=e^{3x} \rightarrow y'=\dfrac{dy}{dx}=3e^{3x}=3y \rightarrow y''=\dfrac{d^2y}{dx^2}=9e^{3x}=9y$

Therefore $y''-6y'+9y=\left[9-18+9\right]y=0$

$y=xe^{3x} \rightarrow y'=e^{3x}+3xe^{3x} \rightarrow y''=6e^{3x}+9xe^{3x}$

Therefore $y''-6y'+9y=0=\left[6+9x-6-18x+9x\right]e^{3x}=0$

3. Show that $\cos 2x$ and $\sin 2x$ are particular solutions of the differential equation $y''+4y=0$.

$y=\cos 2x \rightarrow y'=-2\sin 2x \rightarrow y''=-4\cos 2x=-4y$

$y''+4y=0=\left[-4+4\right]y=0$

and $y=\sin 2x \rightarrow y'=2\cos 2x \rightarrow y''=-4\sin 2x=-4y$

$y''+4y=0=\left[-4+4\right]y=0$

Write down the general solution of the differential equation in

4. Exercise 1: $y = ae^{-2x} + be^{2x/3}$

5. Exercise 2: $y = ae^{3x} + bxe^{3x} = (a + bx)e^{3x}$

6. Exercise 3: $y = a\cos 2x + b\sin 2x$

Section 12.3

Find the general solutions of the differential equations:

7. $y'' - y' - 6y = 0$

The characteristic equation of the differential equation is

$$\begin{aligned}\lambda^2 - \lambda - 6 &= (\lambda - 3)(\lambda + 2)\\ &= 0 \text{ when } \lambda = 3 \text{ and } \lambda = -2\end{aligned}$$

Two particular solutions of the differential equation are therefore

$$y_1 = e^{3x}, \qquad y_2 = e^{-2x}$$

and, because these functions are linearly independent, the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 e^{-2x}$$

8. $2y'' - 8y' + 3y = 0$

The characteristic equation is

$$2\lambda^2 - 8\lambda + 3 = 0 \text{ when } \lambda = \frac{8 \pm \sqrt{64 - 24}}{4} = 2 \pm 2\sqrt{5}$$

The general solution of the differential equation is therefore

$$y = c_1 e^{(2+2\sqrt{5})x} + c_2 e^{(2-2\sqrt{5})x} = e^{2x}\left[c_1 e^{2\sqrt{5}x} + c_2 e^{-2\sqrt{5}x}\right]$$

9. $y'' - 8y' + 16y = 0$

The characteristic equation

$$\lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0$$

has the double root $\lambda = 4$. Two particular solutions are therefore e^{4x} and xe^{4x}, and the general solution is

$$y(x) = (c_1 + c_2 x)e^{4x}$$

10. $4y'' + 12y + 9y = 0$

The characteristic equation

$$4\lambda^2 + 12\lambda + 9 = (2\lambda + 3)^2$$
$$= 0 \text{ when } x = -3/2 \text{ (double root)}$$

The general solution of the differential equation is therefore

$$y(x) = (c_1 + c_2 x)e^{-3x/2}$$

11. $y'' + 4y' + 5y = 0$

The characteristic equation is

$$\lambda^2 + 4\lambda + 5 = 0$$

with roots $\quad \lambda = \frac{1}{2}\left(-4 \pm \sqrt{16 - 20}\right) = -2 \pm i$

The two particular solutions,

$$y_1(x) = e^{(-2+i)x} \quad \text{and} \quad y_2(x) = e^{(-2-i)x}$$

are linearly independent, and the general solution is

$$y(x) = c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x}$$
$$= e^{-2x}(c_1 e^{ix} + c_2 e^{-ix})$$

The equivalent trigonometric form is

$$y(x) = e^{-2x}(a\cos x + b\sin x)$$

12. $y'' + 3y' + 5y = 0$

The characteristic equation

$$\lambda^2 + 3\lambda + 5 = 0$$

has complex roots

$$\lambda = \frac{1}{2}\left(-3 \pm \sqrt{9 - 20}\right) = -\frac{3}{2} \pm \frac{\sqrt{11}}{2} i$$

Then $\quad y(x) = e^{-3x/2}\left[ae^{i\sqrt{11}x/2} + be^{-i\sqrt{11}x/2}\right]$

$$= e^{-3x/2}\left[A\cos\sqrt{11}x/2 + B\sin\sqrt{11}x/2\right]$$

20. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-2y=0;\quad y(0)=2,\ y\to 0$ as $x\to\infty$

As in Exercise 13, the general solution is

$$y(x)=ae^{x}+be^{-2x}$$

The first boundary condition gives

$$y(0)=2=a+b$$

The second condition requires that the solution go to zero as x goes to infinity. The function e^{-2x} has this property but the function e^{x} must be excluded. The condition therefore requires that we set $a=0$.

Then $b=2$ and the solution of the boundary value problem is

$$y(x)=2e^{-2x}$$

21. Solve $\dfrac{d^2\theta}{dt^2}+a^2\theta=0$ subject to the condition $\theta(t+2\pi\tau)=\theta(t)$.

The general solution of the differential equation is

$$\theta(t)=Ae^{iat}+Be^{-iat}.$$

Application of the cyclic boundary condition gives

$$\begin{aligned}\theta(t+2\pi\tau)&=Ae^{ia(t+2\pi\tau)}+Be^{-ia(t+2\pi\tau)}\\&=Ae^{iat}\times e^{i2\pi a\tau}+Be^{-iat}\times e^{-i2\pi a\tau}\\&=\theta(t)\ \text{when}\ e^{\pm i2\pi a\tau}=1\end{aligned}$$

and the condition is satisfied when $2\pi a\tau=2\pi n$ for integer n. Therefore, $a=n/\tau$ and

$$\theta(t)=Ae^{int/\tau}+Be^{-int/\tau},\qquad n=0,\pm1,\pm2,\cdots$$

D1

(7)

$$\frac{dy}{dx} = 2x$$

$$\Leftrightarrow \int dy = \int 2x\,dx$$

$$\Leftrightarrow y = x^2 + c$$

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(15)

$$\frac{dy}{dt} = \frac{e^t}{4y}$$

$$\Leftrightarrow \int 4y\,dy = \int e^t\,dt$$

$$\Leftrightarrow 2y^2 = e^t + c$$

$$\Leftrightarrow y = \pm\sqrt{\tfrac{1}{2}e^t + \tilde{c}}$$

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(18)

$$\frac{dy}{dx} = \sqrt{\frac{x}{y}}$$

$$\Leftrightarrow \int \sqrt{y}\,dy = \sqrt{x}\,dx$$

$$\Leftrightarrow \frac{2}{3}y^{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}} + c$$

$$\Leftrightarrow y = \sqrt[3]{(x^{3/2} + \tilde{c})^2}$$

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(27) $yy' - e^x = 0$ en $y(0) = 4$

$$\Leftrightarrow y\frac{dy}{dx} = e^x$$

$$\Leftrightarrow \int y\,dy = e^x\,dx$$

→

$$\Leftrightarrow \quad \frac{1}{2}y^2 = e^x + c$$

$$\Leftrightarrow \quad y = \pm\sqrt{2e^x + \tilde{c}}$$

$$y(0) = 4 \Rightarrow \quad 4 = +\sqrt{2e^0 + \tilde{c}}$$

$$\uparrow !$$

$$\Leftrightarrow \quad 4 = \sqrt{2 + \tilde{c}}$$

$$\Leftrightarrow \quad 16 = 2 + \tilde{c}$$

$$\Leftrightarrow \quad \tilde{c} = 14$$

$$\Rightarrow \quad y = \sqrt{2e^x + 14}$$

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29 $x(y+4) + y' = 0$ en $y(0) = -5$

$$\Leftrightarrow \quad \frac{dy}{dx} = -x(y+4)$$

$$\Leftrightarrow \quad \int \frac{dy}{y+4} = \int -x\,dx$$

$$\Leftrightarrow \quad \ln(y+4) = -\frac{1}{2}x^2 + c$$

$$\Leftrightarrow \quad y = -4 + e^{-\frac{1}{2}x^2} \cdot \tilde{c}$$

$$y(0) = -5 = -4 + e^0 \cdot \tilde{c} \Rightarrow \tilde{c} = -1$$

$$\Rightarrow \quad y = -4 - e^{-\frac{1}{2}x^2}$$

Opgave **D2**

(1) $y(x) = \frac{1}{2}\left(e^{-3x} + e^{x}\right)$

(2) $y(x) = (2 + 11x)\, e^{-5x}$

(3) $y(x) = \frac{1}{2} \sin(2x)$

Opgave **D3**

$\varepsilon y'' - y' = 0 \Rightarrow \varepsilon v' - v = 0 \Rightarrow v(x) = A e^{\frac{x}{\varepsilon}} \Rightarrow$

$y(x) = \int v(x)\, dx = A \varepsilon e^{\frac{x}{\varepsilon}} + C$

Gebruik randvoorwaarden om A en C te bepalen

Oplossing: $y(x) = \dfrac{e^{\frac{x}{\varepsilon}} - 1}{e^{\frac{1}{\varepsilon}} - 1}$.